Announcements

- Welcome to the class!
- Weekly Office Hours: Wednesdays, 2:40-3:40pm, 10 minutes after class ends on Wednesdays.
- Class web page is at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)
This course will serve as an introduction to submodular functions including methods for their optimization, and how they have been (and can be) applied in many application domains.
Rough Outline

- Introduction to submodular functions, including definitions, real-world and contrived examples of submodular functions, properties, operations that preserve submodularity, submodular variants and special submodular functions, and computational properties.

- Background on submodular functions, including a brief overview of the theory of matroids and lattices.

- Polyhedral properties of submodular functions

- The Lovász extension of submodular functions. The Choquet integral.

- Submodular maximization algorithms under simple constraints, submodular cover problems, greedy algorithms, approximation guarantees
Rough Outline (cont. II)

- Submodular minimization algorithms, a history of submodular minimization, including both numerical and combinatorial algorithms, computational properties of these algorithms, and descriptions of both known results and currently open problems in this area.

- Submodular flow problems, the principle partition of a submodular function and its variants.

- Constrained optimization problems with submodular functions, including maximization and minimization problems with various constraints. An overview of recent problems addressed in the community.

- Applications of submodularity in computer vision, constraint satisfaction, game theory, information theory, norms, natural language processing, graphical models, and machine learning.
Useful Books

- Fujishige, “Submodular Functions and Optimization”, 2005
- Narayanan, “Submodular Functions and Electrical Networks”, 1997
- Schrijver, “Combinatorial Optimization”, 2003
- Additional readings that will be announced here.
Facts about the class

- Prerequisites: ideally knowledge in probability, statistics, convex optimization, and combinatorial optimization these will be reviewed as necessary. The course is open to students in all UW departments. Any questions, please contact me.

- Text: We will be drawing from the book by Satoru Fujishige entitled “Submodular Functions and Optimization” 2nd Edition, 2005, but we will also be reading research papers that will be posted here on this web page, especially for some of the application areas.

- Grades and Assignments: Grades will be based on a combination of a final project (35%), homeworks (45%), and the take home midterm exam (20%). There will be between 3-6 homeworks during the quarter.

- Final project: The final project will consist of a 4-page paper (conference style) and a final project presentation. The project must involve using/dealing mathematically with submodularity in some way or another.
Facts about the class

- Homework/midterm must be submitted electronically using our dropbox (https://catalyst.uw.edu/collectit/dropbox/bilmes/23873). PDF submissions only please. Photos of neatly hand written solutions, combined into one PDF, are fine.

- Lecture slides - are being prepared as we speak. I will try to have them up on the web page the night before each class. I will not only draw from the book but other sources which will be listed at the end of each set of slides.

- Slides from previous version of this class are at http://ssli.ee.washington.edu/~bilmes/ee595a_spring_2011/.
Other logistics

- Almost all equations will have numbers.

By the way, $V \setminus A \equiv \{v \in V : v \not\in A\}$ is set subtraction, sometimes written as $V - A$.
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Theorem 1.1.1 (foo’s theorem)

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**Theorem 1.1.1 (foo’s theorem)**

*foo*

- Exception to these rules is in the review sections, where theorems, equation, etc. (even if repeated) will have new reference numbers.
Outstanding Reading

- Read chapter 1 from Fujishige book.
Announcements, Assignments, and Reminders

- Please do use our discussion board
  (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.
Review

- This is where each day we will be reviewing previous lecture material.
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Discrete Optimization Problems

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**Prof. Jeff Bilmes**

EE596A/Fall 2012/Submodularity – Lecture 1 - September 26th, 2012
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- The general discrete optimization problem we consider here is:

\[
\maximize_{S \subseteq 2^V} f(S) \\
\text{subject to } S \in S
\] (1.3)
Ignoring how complex and general this problem can be for the moment, let's consider some possible applications.

In the rest of this section of slides, we will see many seemingly different applications that, by the end of this course, you will all hopefully see are strongly related to submodularity.

I.e. submodularity is ubiquitous for discrete problems as is convexity for continuous problems.
We are given a finite set $V$ of $n$ elements and a set of subsets $\mathcal{V} = \{V_1, V_2, \ldots, V_m\}$ of $m$ subsets of $V$, so that $V_i \subseteq V$ and $\bigcup_i V_i = V$. 

SET COVER and MAXIMUM COVERAGE
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- The goal in **Maximum Coverage** is, given an integer $k \leq m$, select $k$ subsets, say $\{a_1, a_2, \ldots, a_k\}$ with $a_i \in [m]$ such that $|\bigcup_{i=1}^k V_{a_i}|$ is maximized.
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Both **set cover** and **maximum coverage** are well known to be NP-hard, but have a fast greedy approximation algorithm.
**Definition 1.3.1**

vertex cover A vertex cover (an “vertex-based cover of edges”) in graph $G = (V, E)$ is a set $S \subseteq V(G)$ of vertices such that every edge in $G$ is incident to at least one vertex in $S$.

- Let $I(S)$ be the number of edges incident to vertex set $S$. Then we wish to find the smallest set $S \subseteq V$ subject to $I(S) = |E|$. 

**Definition 1.3.2**

edge cover A edge cover (an “edge-based cover of vertices”) in graph $G = (V, E)$ is a set $F \subseteq E(G)$ of edges such that every vertex in $G$ is incident to at least one edge in $F$.

- Let $|V|(F)$ be the number of vertices incident to edge set $F$. Then we wish to find the smallest set $F \subseteq E$ subject to $|V|(F) = |V|$. 

Sensor Placement

- Given an environment, there is a set $V$ of candidate locations for placement of a sensor (e.g., temperature, gas, audio, video, bacteria or other environmental contaminant, etc.).
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Environment could be a floor of a building, water network, monitored ecological preservation.
An example of a room layout. Should possible to determine temperature at all points in the room. Sensors cannot sense beyond wall (thick black line) boundaries.
Example sensor placement using small range cheap sensors (located at red dots).
Sensor Placement in Buildings

- Example sensor placement using large range expensive sensors (located at red dots).
Sensor Placement in Buildings

- Example sensor placement using mixed range sensors (located at red dots).
Graph Cut Problems

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- Weighted versions.
- Many examples of this, we will see more later.
Facility/Plant Location (uncapacitated)

- Core problem in operations research and strong original motivation for submodular functions.
- Goal: as efficiently as possible, place “facilities” (factories) at certain locations to satisfy sites (at all locations) having various demands.
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```
  1  2  3  4  5
facility locations

  1  2  3  4  5
sites
```

Graph representation:

- Facility locations: 1, 2, 3, 4, 5
- Sites: 1, 2, 3, 4, s
- Edge: C_{2A}

Prof. Jeff Bilmes
Facility/Plant Location (uncapacitated)

- Let $F = \{1, \ldots, f\}$ be a set of possible factory/plant locations for facilities to be built.
- $S = \{1, \ldots, s\}$ is a set of sites needing to be serviced (e.g., cities, clients).
- Let $c_{ij}$ be the “benefit” (e.g., $1/c_{ij}$ is the cost) of servicing site $i$ with facility location $j$.
- Let $m_j$ be the benefit (e.g., either $1/m_j$ is the cost or $-m_j$ is the cost) to build a plant at location $j$.
- Each site needs to be serviced by only one plant but no less than one.
- Define $f(A)$ as the “delivery benefit” plus “construction benefit” when the locations $A \subseteq F$ are to be constructed.
- We can define $f(A) = \sum_{j \in A} m_j + \sum_{i \in F} \max_{j \in A} c_{ij}$.
- Goal is to find a set $A$ that maximizes $f(A)$ (the benefit) placing a bound on the number of plants $A$ (e.g., $|A| \leq k$).
Information Gain and Feature Selection

- Task: pattern recognition based on (at most) features $X_V$ to predict random variable $Y$. True model is $p(Y, X_V)$, where $V$ is a finite set of feature indices.
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$$f(A) = I(Y; X_A) = H(Y) - H(Y|X_A) = H(X_A) - H(X_A|Y) \quad (1.4)$$
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Applicable not only in pattern recognition, but in the sensor coverage problem as well, where $Y$ is whatever question we wish to ask about the room.
Monge Matrices

- $m \times n$ matrices $C = [c_{ij}]_{ij}$ are called Monge matrices if they satisfy the Monge property, namely:

\[
c_{ij} + c_{rs} \leq c_{is} + c_{rj} \tag{1.5}
\]

for all $1 \leq i < r \leq m$ and $1 \leq j < s \leq n$. 

Useful for speeding up certain dynamic programming problems.
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Monge Matrices

- Can generate a Monge matrix from a convex polygon - delete two segments, separately number vertices on each chain, distances $c_{ij}$ satisfy Monge property (or quadrangle inequality).
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A model of Influence in Social Networks

- Given a graph $G = (V, E)$, each $v \in V$ corresponds to a person, to each $v$ we have an activation function $f_v : 2^V \rightarrow [0, 1]$ dependent only on its neighbors. I.e., $f_v(A) = f_v(A \cap \Gamma(v))$. 

Goal - Viral Marketing: find a small subset $S \subseteq V$ of individuals to directly influence, and thus indirectly influence the greatest number of possible other individuals (via the social network $G$).
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We define a function $f : 2^V \to \mathbb{Z}^+$ that models the ultimate influence of an initial set $S$ of nodes based on the following iterative process: At each step, a given set of nodes $S$ are activated, and we activate new nodes $v \in V \setminus S$ if $f_v(S) \geq U[0, 1]$ (where $U[0, 1]$ is a uniform random number between 0 and 1).
Let $V$ be a group of individuals. How valuable to you is a given friend $v \in V$?
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It depends on how many friends you have.
The value of a friend

- Let $V$ be a group of individuals. How valuable to you is a given friend $v \in V$?
- It depends on how many friends you have.
- Given a group of friends $S \subseteq V$, can you valuate them with a function $f(S)$ and how?
Let $V$ be a group of individuals. How valuable to you is a given friend $v \in V$?

It depends on how many friends you have.

Given a group of friends $S \subseteq V$, can you valuate them with a function $f(S)$ an how?

Let $f(S)$ be the value of the set of friends $S$. Is submodular or supermodular a good model?
Information Cascades

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![Diagram of information cascade network]
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- How to model flow of information from source to the point it reaches users — information used in its common sense (like news events).
- How to find the most influential sources, the ones that often set off cascades, which are like large “waves” of information flow?
- There can be one seed source (shown below) or many.
Given a set $V$ of items, how do we choose a subset $S \subseteq V$ that is as diverse as possible, with perhaps constraints on $S$ such as its size.
Diversity Functions

- Given a set $V$ of items, how do we choose a subset $S \subseteq V$ that is as diverse as possible, with perhaps constraints on $S$ such as its size.
- How do we choose the smallest set $S$ that maintains a given quality of diversity?
Motivation & Applications

Basic Definitions

Summary

Scratch

Diversity Functions

- Given a set $V$ of items, how do we choose a subset $S \subseteq V$ that is as diverse as possible, with perhaps constraints on $S$ such as its size.
- How do we choose the smallest set $S$ that maintains a given quality of diversity?
- Goal of diversity: ensure proper representation in chosen set that, say otherwise in a random sample, would lead to poor representation of normally underrepresented groups.
Given a graphical model $G = (V, E)$ and a family of probability distributions $p \in \mathcal{F}(G, \mathcal{M})$ that factor w.r.t. that distribution.
Graphical Model Tree Distributions

- Given a graphical model $G = (V, E)$ and a family of probability distributions $p \in \mathcal{F}(G, \mathcal{M})$ that factor w.r.t. that distribution.
- Find the closest distribution $p_t$ to $p$ subject to $p_t$ factoring w.r.t. some tree $T = (V, F)$, i.e., $p_t \in \mathcal{F}(T, \mathcal{M})$. 
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- I.e., optimization problem

$$\min_{p_t \in \mathcal{F}(G, \mathcal{M})} D(p || p_t)$$

subject to

$$p_t \in \mathcal{F}(T, \mathcal{M}).$$

$$T = (V, F') \text{ is a tree}$$

(1.6)
Graphical Model Tree Distributions

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$$
\begin{align*}
\text{minimize} & \quad D(p \| p_t) \\
\text{subject to} & \quad p_t \in \mathcal{F}(T, \mathcal{M}) \\
& \quad T = (V, F') \text{ is a tree}
\end{align*}
$$

- Discrete problem: Choose the right subset of edges from $E$ that make up a tree (i.e., find a spanning tree of $G$ of best quality).
The figure below represents the sentences of a document
We extract sentences (green) as a summary of the full document.

The summary on the left is a subset of the summary on the right. Consider adding a new (blue) sentence to each of the two summaries. The marginal (incremental) benefit of adding the new (blue) sentence to the smaller (left) summary is no more than the marginal benefit of adding the new sentence to the larger (right) summary.

---

**Prof. Jeff Bilmes**

EE596A/Fall 2012/Submodularity – Lecture 1 - September 26th, 2012

page 1-33 (of 177)
Extractive Document Summarization

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Extractive Document Summarization

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- **diminishing returns ↔ submodularity**
A web search is a form of summarization based on query.

Goal of a web search engine is to produce a ranked list of web pages that, conditioned on the text query entered, summarizes the most important links on the web.

Information retrieval (the science of automatically acquiring information), book and music recommendation systems —

Overall goal: user should quickly find information that is informative, concise, accurate, relevant (to the user’s needs), and comprehensive.
Active Learning and Semi-Supervised Learning

- Given training data $\mathcal{D}_V = \{(x_i, y_i)\}_{i \in V}$ of $(x, y)$ pairs where $x$ is a query (data item) and $y$ is an answer (label), goal is to learn a good mapping $y = h(x)$. 
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- Often, getting $y$ is time-consuming, expensive, and error prone (manual transcription, Amazon Turk, etc.)
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Batch active learning: choose a subset \(S \subset V\) so that only the labels \(\{y_i\}_{i \in S}\) should be acquired.
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- Adaptive active learning: choose a policy whereby we choose an $i_1 \in V$, get the label $y_{i_1}$, choose another $i_2 \in V$, get label $y_{i_2}$, where each choose can be based on previously acquired labels.
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- Semi-supervised (transductive) learning: Once we have $\{y_i\}_{i \in S}$, infer the remaining labels $\{y_i\}_{i \in V \setminus S}$. 
Economies of Scale: Many goods and services can be produced at a much lower per-unit cost only if they are produced in very large quantities.
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If you already make a good, making a similar good is easier than if you start from scratch (e.g., Apple making both iPod and iPhone).
Markets: Supply Side Economies of Scale

- **Economies of Scale**: Many goods and services can be produced at a much lower per-unit cost only if they are produced in very large quantities.
- The profit margin for producing a unit of goods improved as more of those goods are created.
- If you already make a good, making a similar good is easier than if you start from scratch (e.g., Apple making both iPod and iPhone).
- An argument in favor of free trade is that it opens up larger markets to firms in (especially otherwise small markets), thereby enabling better economies of scale, and hence greater efficiency (lower costs and resources per unit of good produced).
Supply Side Economies of scale: Cost of manufacturing a set of items

- Let $V$ be a set of possible items that a company might possibly wish to manufacture, and let $f(S)$ for $S \subseteq V$ be the cost to that company to manufacture subset $S$. 
Supply Side Economies of scale: Cost of manufacturing a set of items

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- Ex: $V$ might be colors of paint in a paint manufacturer: green, red, blue, yellow, white, etc.
Supply Side Economies of scale: Cost of manufacturing a set of items

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- Ex: $V$ might be colors of paint in a paint manufacturer: green, red, blue, yellow, white, etc.
- Producing green when you are already producing yellow and blue is probably cheaper than if you were only producing some other colors.

$$f(\text{green, blue, yellow}) - f(\text{blue, yellow}) \leq f(\text{green, blue}) - f(\text{blue})$$  \hspace{1cm} (1.7)
Demand side Economies of Scale: Network Externalities

- consumers of a good derive positive value when size of the market increases.
**Demand side Economies of Scale: Network Externalities**

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- The value of a network to a user depends on the number of other users in that network.
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- Given network externalities, a consumer in today’s market cares also about the future success of the product and competing products.
- If the good is durable (or there is human capital investment), the total benefits derived from a good will depend on the number of consumers who adopt compatible products in the future.
Positive Network Externalities

- railroad - standard rail format and shared access
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- The telephone, who wants to talk by phone only to oneself?
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- social network sites: facebook more valuable with everyone online

Concepts like the “tipping point”, and “winner take all” markets.
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No Network Externalities

- food/drink - (should be) independent of how many others are eating the type of food.
Other Network Externalities

No Network Externalities

- food/drink - (should be) independent of how many others are eating the type of food.
- Music - your enjoyment should be independent of others’ enjoyment.
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Negative Network Externalities

- clothing
Other Network Externalities

No Network Externalities

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- Music - your enjoyment should be independent of others’ enjoyment.

Negative Network Externalities

- clothing
- costumes
Optimization Problem Involving Network Externalities

(From Mirrokni, Roch, Sundararajan 2012): Let $V$ be a set of users.
Optimization Problem Involving Network Externalities

- (From Mirrokni, Roch, Sundararajan 2012): Let $V$ be a set of users.
- Let $v_i(S)$ be the value that user $i$ has for a good if $S \subseteq V$ already own the good — e.g. $v_i(S) = \omega_i + f_i(\sum_{j \in S} w_{ij})$ where $\omega_i$ is inherent value, and $f_i$ might be a concave function, and $w_{ij}$ is now important $j \in S$ is to $i$ (e.g., a network).
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- Given price $p$ for good, user $i$ buys good if $v_i(S) \geq p$. 
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- Given price $p$ for good, user $i$ buys good if $v_i(S) \geq p$.
- We choose initial price $p$ and initial set of users $A \subseteq V$ who get the good for free.
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Define $S_1 = \{i \notin A : v_i(A) \geq p\}$ initial set of buyers.
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$S_2 = \{i \notin A \cup S_1 : v_i(A \cup S_1) \geq p\}$. 
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This starts a cascade. Let $S_k = \bigcup_{j<k} S_j \cup A | v_j(\bigcup_{j<k} S_j \cup A) \geq p\},$
Optimization Problem Involving Network Externalities

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\( S_2 = \{ i \notin A \cup S_1 : v_i(A \cup S_1) \geq p \} \).

This starts a cascade. Let \( S_k = \{ \cup_{j<k} S_j \cup A | v_j(\cup_{j<k} S_j \cup A) \geq p \} \),

and let \( S_k^* \) be the saturation point, lowest value of \( k \) such that \( S_k = S_{k+1} \).
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and let $S_k^*$ be the saturation point, lowest value of $k$ such that $S_k = S_{k+1}$

Goal: find $A$ and $p$ to maximize $p \times |S_k^*|$. 
Image Segmentation

- an image needing to be segmented.
labeled data in the form of some pixels being marked foreground (red) and others being marked background (blue).
Image Segmentation

- the foreground is removed from the background.
Markov random field

$$\log p(x) \propto \sum_{v \in V(G)} e_v(x_v) + \sum_{(i,j) \in E(G)} e_{ij}(x_i, x_j)$$  \hspace{1cm} (1.8)

When $G$ is a 2D grid graph, we have
Markov random fields and image segmentation

- We can create auxiliary graph that involves two new nodes $s$ and $t$ and connect each of $s$ and $t$ to all of the original nodes.
- I.e., $G_a = (V \cup \{s, t\}, E + \cup_{v \in V} ((s, v) \cup (v, t)))$. 
Markov random fields and image segmentation

Original Graph: \( \log p(x) \propto \sum_{v \in V(G)} e_v(x_v) + \sum_{(i,j) \in E(G)} e_{ij}(x_i, x_j) \)
Augmented graph-cut graph. The edge weights of graph are derived from \( \{e_v\}_{v \in V} \) and \( \{e_{ij}\}_{(i,j) \in E(G)} \).
Augmented graph-cut graph with indicated cut corresponding to particular vector $\bar{x} \in \{0, 1\}^n$. Each cut $\bar{x}$ has a score corresponding to $p(\bar{x})$. 
Other applications in or related to computer vision

- Image denoising
- Multi-label graph cuts
- Graphical model inference, computing the Viterbi (or the MPE or the MAP) assignment of a set of random variables.
- Clustering of data
Given a set of random variables \( \{X_i\}_{i \in V} \) indexed by set \( V \), how do we partition them so that we can best block-code them within each block.
Information Theory

- Given a set of random variables $\{X_i\}_{i \in V}$ indexed by set $V$, how do we partition them so that we can best block-code them within each block.
- I.e., how do we form $S \subseteq V$ such that $I(X_S; X_{V \setminus S})$ is as small as possible, where $I(X_A; X_B)$ is the mutual information between random variables $X_A$ and $X_B$. 
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A network of senders/receivers
A network of senders/receivers

Each sender $X_i$ is trying to communicate simultaneously with each receiver $Y_i$ (i.e., for all $i$, $X_i$ is sending to $\{Y_i\}_i$
A network of senders/receivers

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I.e., can we find functions $f$ such that any rates must satisfy

$$\forall S \subseteq V, \sum_{i \in S, j \in V \setminus S} R^{(i\rightarrow j)} \leq f(S) \quad (1.9)$$
Anecdote

From David Brooks, NYTs column, March 28th, 2011 on “Tools for Thinking”. In response to Steven Pinker (Harvard) asking a number of people “What scientific concept would improve everybody’s cognitive toolkit?”

Emergent systems are ones in which many different elements interact. The pattern of interaction then produces a new element that is greater than the sum of the parts, which then exercises a top-down influence on the constituent elements.
Given a set of objects $V = \{v_1, \ldots, v_n\}$ and a function $f : 2^V \to \mathbb{R}$ that returns a real value for any subset $S \subseteq V$.

Suppose we are interested in finding the subset that either maximizes or minimizes the function, e.g., $\arg\max_{S \subseteq V} f(S)$, possibly subject to some constraints.

In general, this problem has exponential time complexity.

Example: $f$ might correspond to the value (e.g., information gain) of a set of sensor locations in an environment, and we wish to find the best set $S \subseteq V$ of sensors locations given a fixed upper limit on the number of sensors $|S|$.

In many cases (such as above) $f$ has properties that make its optimization tractable to either exactly or approximately compute.

One such property is submodularity.
Definition 1.4.1 (submodular concave)

A function \( f : 2^V \to \mathbb{R} \) is submodular if for any \( A, B \subseteq V \), we have that:

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B)
\]  

(1.10)

An alternate and (as we see in lecture 3) equivalent definition is:

Definition 1.4.2 (diminishing returns)

A function \( f : 2^V \to \mathbb{R} \) is submodular if for any \( A \subseteq B \subset V \), and \( v \in V \setminus B \), we have that:

\[
f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)
\]  

(1.11)

This means that the incremental “value”, “gain”, or “cost” of \( v \) decreases (diminishes) as the context in which \( v \) is considered grows from \( A \) to \( B \).
Definition 1.4.3 (subadditive)

A function $f : 2^V \rightarrow \mathbb{R}$ is subadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) \quad (1.12)$$

This means that the “whole” is less than the sum of the parts.
Supermodular Definitions

Definition 1.4.4 (supermodular convex)

A function $f : 2^V \to \mathbb{R}$ is supermodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \leq f(A \cup B) + f(A \cap B) \quad (1.13)$$

An alternate and equivalent definition is:

Definition 1.4.5 (increasing returns)

A function $f : 2^V \to \mathbb{R}$ is supermodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \leq f(B \cup \{v\}) - f(B) \quad (1.14)$$

The incremental “value”, “gain”, or “cost” of $v$ increases as the context in which $v$ is considered grows from $A$ to $B$. 
Definition 1.4.6 (superadditive)

A function \( f : 2^V \rightarrow \mathbb{R} \) is superadditive if for any \( A, B \subseteq V \), we have that:

\[
f(A) + f(B) \leq f(A \cup B) \tag{1.15}
\]

This means that the “whole” is greater than the sum of the parts.
**Definition 1.4.7 (modular)**

A function that is both submodular and supermodular is called **modular**.

If $f$ is a modular function, then for any $A, B \subseteq V$, we have

$$f(A) + f(B) = f(A \cap B) + f(A \cup B)$$  \hspace{1cm} (1.16)

Modular functions have no interaction, and have value based only on singleton values.

**Proposition 1.4.8**

*If $f$ is modular, it may be written as*

$$f(A) = f(\emptyset) + \sum_{a \in A} \left( f(\{a\}) - f(\emptyset) \right)$$  \hspace{1cm} (1.17)
Proof.

We inductively construct the value for \( A = \{a_1, a_2, \ldots, a_k\} \).

\[
f(a_1) + f(a_2) = f(a_1, a_2) + f(\emptyset) \tag{1.18}
\]

implies

\[
f(a_1, a_2) = f(a_1) - f(\emptyset) + f(a_2) - f(\emptyset) + f(\emptyset) \tag{1.19}
\]

then

\[
f(a_1, a_2) + f(a_3) = f(a_1, a_2, a_3) + f(\emptyset) \tag{1.20}
\]

implies

\[
f(a_1, a_2, a_3) = f(a_1, a_2) - f(\emptyset) + f(a_3) - f(\emptyset) + f(\emptyset) \tag{1.21}
\]

\[
= f(\emptyset) + \sum_{i=1}^{3} f(a_i) - f(\emptyset) \tag{1.22}
\]
Complement function

Given a function $f : 2^V \to \mathbb{R}$, we can find a complement function $\bar{f} : 2^V \to \mathbb{R}$ as $\bar{f}(A) = f(V \setminus A)$ for any $A$.

**Proposition 1.4.9**

$\bar{f}$ is submodular if $f$ is submodular.

**Proof.**

\[
\bar{f}(A) + \bar{f}(B) \geq \bar{f}(A \cup B) + \bar{f}(A \cap B) \tag{1.23}
\]

follows from

\[
f(V \setminus A) + f(V \setminus B) \geq f(V \setminus (A \cup B)) + f(V \setminus (A \cap B)) \tag{1.24}
\]

which is true because $V \setminus (A \cup B) = (V \setminus A) \cap (V \setminus B)$ and $V \setminus (A \cap B) = (V \setminus A) \cup (V \setminus B)$. 

Submodularity

- Submodular functions have a long history in economics, game theory, combinatorial optimization, electrical networks, and operations research.
- They are gaining importance in machine learning as well (one of our main motivations for offering this course).
- Arbitrary set functions are hopelessly difficult to optimize, while the minimum of submodular functions can be found in polynomial time, and the maximum can be constant-factor approximated in low-order polynomial time.
- Submodular functions share properties in common with both convex and concave functions.
Attractions of Convex Functions

Why do we like Convex Functions? (Quoting Lovász 1983):

1. Convex functions occur in many mathematical models in economy, engineering, and other sciences. Convexity is a very natural property of various functions and domains occurring in such models; quite often the only non-trivial property which can be stated in general.
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3. Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.

4. There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.
In this course, we wish to demonstrate that submodular functions also possess attractions of these four sorts as well.