Logistics

Review

Announcements, Assignments, and Reminders

- Please do use our discussion board (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.

- I’ll likely be gone the week of Dec 3rd-7th. Hence, we’ll most likely have two extra lectures the following week (during finals week). More details TBA.
Cumulative Outstanding Reading

- Read Tom McCormick's overview paper on SFM [http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf](http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf)
- Read chapters 1 - 3 from Fujishige book.
- Read over lecture slides, all posted on our web page ([http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/](http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)).
- See the summary slide at the end for lectures for additional reading sources.
First, a bit on $D(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$D(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \}$$

(10.9)

Theorem 10.2.6

For any $y \in P_f^+$, with $f$ a polymatroid function, then $D(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 10.3.6

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_E^+$.  

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in D(y) \}$$

(10.10)

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting?
- Sort elements of $E$ w.r.t. $w$ so that, w.l.o.g.  
  $$E = (e_1, e_2, \ldots, e_m) \text{ with } w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).$$
- Let $k + 1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
  
  That is, we have
  $$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_m)$$
  (10.1)

- Next define partial accumulated sets $E_i$, for $i = 0 \ldots m$, we have w.r.t. the above sorted order:
  $$E_i \overset{\text{def}}{=} \{ e_1, e_2, \ldots, e_i \}$$
  (10.3)

  (note $E_0 = \emptyset$, $f(E_0) = 0$, and $E$ and $E_i$ is always sorted w.r.t $w$).
- The greedy solution is the vector $x \in \mathbb{R}_E^+$ with elements defined as:
  $$x(e_1) \overset{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset)$$
  (10.4)

  $$x(e_i) \overset{\text{def}}{=} 0 \text{ for } i = k + 1 \ldots m = |E|$$
  (10.6)
Theorem 10.2.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes $wx$ over $P_f$.

Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f) = \min \left( \sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A 1_A \geq w \right)$$

(10.1)

- Define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \overset{\text{def}}{=} w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m - 1),$$

(10.2)

$$y_E \overset{\text{def}}{=} w(e_m), \text{ and}$$

(10.3)

$$y_A = 0 \text{ otherwise} \quad (10.4)$$

Conversely, suppose $P_f^+$ is a polytope of form

$$P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\},$$

then the greedy solution to

$$\max(wx : x \in P)$$

is optimum only if $f$ is submodular.

Proof.

- Order elements of $E$ arbitrarily as $(e_1, e_2, \ldots, e_m)$ and define $E_i = (e_1, e_2, \ldots, e_i)$.

- For $1 \leq p \leq q \leq m$, define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p)$.

- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$.

- Define $w \in \{0, 1\}^m$ as:

$$w \overset{\text{def}}{=} \sum_{i=1}^q 1_{e_i} = 1_{A \cup B} \quad (10.1)$$

- Suppose optimum solution $x$ is given by the greedy procedure.
Polymatroidal polyhedron and greedy

Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 7.3.1)

**Theorem 10.2.1**

If \( f : 2^E \to \mathbb{R}_+ \) is given, and \( P \) is a polytope in \( \mathbb{R}_+^E \) of the form

\[
P = \left\{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \right\},
\]

then the greedy solution to the problem \( \max(\mathbf{w}^T x : x \in P) \) is \( \forall \mathbf{w} \) optimum iff \( f \) is monotone non-decreasing submodular (i.e., iff \( P \) is a polymatroid).

Multiple Polytopes associated with arbitrary \( f \)

\[
P_f^+ = P_f \cap \{ x \in \mathbb{R}^E : x \geq 0 \} \quad (10.5)
\]

\[
P_f = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \} \quad (10.6)
\]

\[
B_f = P_f \cap \{ x \in \mathbb{R}^E : x(E) = f(E) \} \quad (10.7)
\]
Polymatroid extreme points

**Theorem 10.2.1**

For a given ordering $E = (e_1, \ldots, e_m)$ of $E$ and a given $E_i$ and $x$ generated by $E_i$ using the greedy procedure, then $x$ is an extreme point of $P_f$

**Proof.**

- We already saw that $x \in P_f$ (Theorem 10.2.1).
- To show that $x$ is an extreme point of $P_f$, note that it is the unique solution of the following system of equations

\[
x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \quad (10.9)
\]

\[
x(e) = 0 \text{ for } e \in E \setminus E_i \quad (10.10)
\]

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the $x$ constructed via the Greedy algorithm!!
Logistics

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A + e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.
Intuition: why greedy works with polymatroids

- Given $w$, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^T w = x(e_1) w(e_1) + x(e_2) w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^T w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^T w$ over $x \in P_f^+$.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A + e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 10.3.6**

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}^E_+$, and any $P_f^+$-basis $y^x \in \mathbb{R}^E$ of $x$, the component sum of $y^x$ is

$$y^x(E) = \text{rank}(x) = \max \left( y(E) : y \leq x, y \in P_f^+ \right)$$

$$= \min \left( x(A) + f(E \setminus A) : A \subseteq E \right) \quad (10.1)$$

As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$.

By taking elements $E \setminus A$ to be zero in $x$, we can define/recover the submodular function from the polymatroid polyhedron via the following:

$$f(A) = \max \left\{ y(A) : y \in P_f^+ \right\} \quad (10.2)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that $P_f^+$ is a polymatroid).
Considering Theorem 10.3.6, the matroid case is now a special case, where we have that:

**Corollary 10.3.8**

We have that:

\[
\max \{ y(E) : y \in P_{\text{ind. set}}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\]

(10.27)

where \( r_M \) is the matroid rank function of some matroid.

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Consider

\[
P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \}
\]

(10.1)

Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \not\in P_r^+ \).

Hence, there must be a set of \( \mathcal{W} \subseteq 2^V \), each member of which corresponds to a violated inequality, i.e., equations of the form \( x(A) > r_M(A) \) for \( A \in \mathcal{W} \).

The most violated inequality when \( x \) is considered w.r.t. \( P_r^+ \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e., the most violated inequality is valued as:

\[
\max \{ x(A) - r_M(A) : A \in \mathcal{W} \} = \max \{ x(A) - r_M(A) : A \subseteq E \}
\]

(10.2)

Since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \), we can express this via a min as in::

\[
\min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\]

(10.3)
Consider

\[ P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \} \]  

(10.4)

Suppose we have any \( x \in \mathbb{R}^E_+ \) such that \( x \not\in P_f^+ \).

Hence, there must be a set of \( \mathcal{W} \subseteq 2^V \), each member of which corresponds to a violated inequality, i.e., equations of the form \( x(A) > r_M(A) \) for \( A \in \mathcal{W} \).

The most violated inequality when \( x \) is considered w.r.t. \( P_f^+ \) corresponds to the set \( A \) that maximizes \( x(A) - f(A) \), i.e., the most violated inequality is valuated as:

\[
\max \{ x(A) - f(A) : A \in \mathcal{W} \} = \max \{ x(A) - f(A) : A \subseteq E \} \tag{10.5}
\]

Since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \), we can express this via a min as in:

\[
\min \{ f(A) + x(E \setminus A) : A \subseteq E \} \tag{10.6}
\]

More importantly, \( \min \{ f(A) + x(E \setminus A) : A \subseteq E \} \) a form of submodular function minimization, namely \( \min \{ f(A) - x(A) : A \subseteq E \} \) for a submodular function consisting of a difference of polymatroid and modular function (so no longer necessarily monotone, nor positive).
Matroids, other definitions using matroid rank $r : 2^V \rightarrow \mathbb{Z}_+$

**Definition 10.4.6 (closed/flat/subspace)**

A subset $A \subseteq E$ is **closed** or a **flat** or a **subspace** of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

**Definition 10.4.7 (closure)**

Given $A \subseteq E$, the **closure** (or span) of $A$, is defined by
definitionn\[##\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}##.\]

Therefore, a closed set $A$ has $\text{span}(A) = A$.

**Definition 10.4.8 (circuit)**

A subset $A \subseteq E$ is **circuit** or a **cycle** if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 10.4.6 (Matroid by circuits)**

Let $E$ be a set and $\mathcal{C}$ be a collection of nonempty subsets of $E$, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. $\mathcal{C}$ is the collection of circuits of a matroid;
2. if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;
3. if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing $y$;

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.
Lemma 10.4.1

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in $M$.

Proof.

- Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit $C_3$ of $M$ s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$.
- This contradicts the independence of $I$.

Define $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (the fundamental circuit in $M$ w.r.t. $I$ and $e$, if it exists).

If $e \in \text{span}(I) \setminus I$, then $C(I, e)$ is well defined ($I + e$ creates one circuit).

If $e \in I$, then $I + e = I$ doesn’t create a circuit. In such cases, $C(I, e)$ is not really defined.

In such cases, we define $C(I, e) = \{e\}$, and we will soon see why.

If $e \notin \text{span}(I)$, then $C(I, e) = \emptyset$, since no circuit is created in this case.
**Lemma 10.4.2**

Let $\mathcal{B}(C)$ be the set of bases of $C$. Then, given matroid $\mathcal{M} = (E, \mathcal{I})$, and any set $C \subseteq E$, we have that:

$$
\bigcup_{B \in \mathcal{B}(C)} B = C.
$$

**(10.7)**

**Proof.**

- Define $C' \triangleq \bigcup_{B \in \mathcal{B}(C)}$, and suppose $\exists c \in C$ such that $c \notin C'$.
- Hence, $\forall B \in \mathcal{B}(C)$ we have $c \not\in B$, and $B + c$ contains a single circuit for any $B$, namely $C(B, c)$.
- Then choose $c' \in C(B, c)$ with $c' \neq c$.
- Then $B + c - c'$ is independent size $|B|$ subset of $C$ and hence spans $C$, and thus is a $c$-containing member of $\mathcal{B}(C)$, contradicting $c \notin C'$.

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**The sat function = Polymatroid Closure**

- Thus, in a matroid, closure (span) of a set $A$ are all items that $A$ spans (eq. that depend on $A$).
- We wish to generalize closure to polymatroids.
- Consider $x \in P_f$ for polymatroid function $f$.
- Again, recall, tight sets are closed under union and intersection, and therefore form a distributive lattice.
- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$, which can constitute a join and meet.
- Recall, for a given $x \in P_f$, we have defined this tight family as

$$
\mathcal{D}(x) = \{ A : A \subseteq E, x(A) = f(A) \}
$$

**(10.8)**
The \( sat \) function = Polymatroid Closure

- Now given \( x \in P_f^+ \):

\[
D(x) = \{ A : A \subseteq E, x(A) = f(A) \} \\
= \{ A : f(A) - x(A) = 0 \} 
\]

(10.9)

(10.10)

- Since \( x \in P_f^+ \) and \( f \) is presumed to be polymatroid function, we see \( f'(A) = f(A) - x(A) \) is a non-negative submodular function, and \( D(x) \) are the zero-valued minimizers (if any) if \( f'(A) \).

- The zero-valued minimizers of \( f' \) are thus closed under union and intersection.

- In fact, this is true for all minimizers of a submodular function as stated in the next theorem.

Minimizers of a Submodular Function form a lattice

**Theorem 10.5.1**

For arbitrary submodular \( f \), the minimizers are closed under union and intersection. That is, let \( M = \arg\min_{X \subseteq E} f(X) \) be the set of minimizers of \( f \). Let \( A, B \in M \). Then \( A \cup B \in M \) and \( A \cap B \in M \).

**Proof.**

Since \( A \) and \( B \) are minimizers, we have \( f(A) = f(B) \leq f(A \cap B) \) and \( f(A) = f(B) \leq f(A \cup B) \).

By submodularity, we have

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B) 
\]

(10.11)

Hence, we must have \( f(A) = f(B) = f(A \cup B) = f(A \cap B) \).

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.
The sat function = Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $D(x)$, also called the polymatroid closure or sat (saturation function).
- For some $x \in P_f$, we have defined:
  \[ \text{cl}(x) \defeq \text{sat}(x) \defeq \bigcup \{ A : A \in D(x) \} \]
  \[ = \bigcup \{ A : A \subseteq E, x(A) = f(A) \} \]
  \[ = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \} \] (10.14)

- Hence, sat($x$) is the maximal minimizer of the submodular function $f_x(A) \triangleq f(A) - x(A)$.
- Eq. (10.14) says that sat consists of any point $x$ that is $P_f$ saturated (any additional positive movement, in that dimension, leaves $P_f$). We’ll revisit this in a few slides.
- First, we see how sat generalizes matroid closure.

Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $1_I \in P_r$ and

\[ D(1_I) = \{ A : 1_I(A) = r(A) \} \] (10.15)

and

\[ \text{sat}(1_I) = \bigcup \{ A : A \subseteq E, A \in D(1_I) \} \] (10.16)
\[ = \bigcup \{ A : A \subseteq E, 1_I(A) = r(A) \} \] (10.17)
\[ = \bigcup \{ A : A \subseteq E, |I \cap A| = r(A) \} \] (10.18)

- Notice that $1_I(A) = |I \cap A|$.
- Intuitively, $|I \cap A| \leq |I|$. Also, consider an $A \supset I \in \mathcal{I}$ that doesn’t increase rank, meaning $r(A) = r(I)$. If $r(A) = |I \cap A|$ then $A$ is in $I$’s span.
- We formalize this next.
Lemma 10.5.2 (Matroid: \( \mathbb{R}^E_+ \to 2^E \) is the same as closure.)

For \( I \in \mathcal{I} \), we have \( \text{sat}(1_I) = \text{span}(I) \) \hspace{1cm} (10.19)

Proof.
- For \( A = I \), \( 1_I(I) = |I| = r(I) \), so \( I \in \mathcal{D}(1_I) \) and \( I \subseteq \text{sat}(1_I) \).
  Also, \( I \subseteq \text{span}(I) \).
- Consider some \( b \in \text{span}(I) \setminus I \).
- Then \( A = I \cup \{b\} \in \mathcal{D}(1_I) \) since \( 1_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I) \).
- Thus, \( b \in \text{sat}(1_I) \).
- Therefore, \( \text{sat}(1_I) \supseteq \text{span}(I) \).

... proof continued.

- Now, consider \( b \in \text{sat}(1_I) \setminus I \).
- Choose any \( A \in \mathcal{D}(1_I) \) with \( b \in A \).
- Then \( |A \cap I| = r(A) \).
- Now \( r(A) = |A \cap I| \leq |I| = r(I) \).
- Also, \( r(A \cap I) = |A \cap I| \) since \( A \cap I \in \mathcal{I} \).
- Hence, \( r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I)) \) meaning \( (A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I) \).
- Since \( b \in A \setminus I \), \( b \in \text{span}(I) \).
- Thus, \( \text{sat}(1_I) \subseteq \text{span}(I) \).
- Hence \( \text{sat}(1_I) = \text{span}(I) \).
The sat function = Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.
- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
- Note $\text{span}(C) = \text{span}(B)$ where $B \subseteq B(C)$ is a base of $C$.
- Then we have $1_B \leq 1_C \leq 1_{\text{span}(C)}$, and that $1_B \in P_r$. We can then make the definition:

$$\text{sat}(1_C) \triangleq \text{sat}(1_B) \text{ for } B \in B(C)$$  \hfill (10.20)

In which case, we also get $\text{sat}(1_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(\text{P-basis}(y))$).
- However, consider the following form

$$\text{sat}(1_C) = \bigcup \{A : A \subseteq E, |A \cap C| = r(A)\}$$  \hfill (10.21)

Exercise: is $\text{span}(C) = \text{sat}(1_C)$? Prove or disprove it.

The sat function, span, and submodular function minimization

- Thus, for a matroid, $\text{sat}(1_I)$ is exactly the closure (or span) of $I$ in the matroid. i.e., for matroid $(E, r)$, we have $\text{span}(I) = \text{sat}(1_B)$.
- Recall, for $x \in P_f$ and polymatroidal $f$, $\text{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, $\text{span}(I)$ is the maximal minimizer of the submodular function formed by $r(A) - 1_I(A)$.
- Submodular function minimization can solve “span” queries in a matroid or “sat” queries in a polymatroid.
- In general, given polymatroid function $f : 2^V \rightarrow \mathbb{R}$, there exists a form of span in that, given $A$, we wish to find the largest set $B$ such that $f(B \cup A) = f(A)$.
- Find largest minimizer of $g : 2^{V \setminus A} \rightarrow \mathbb{R}$ with $g(B) = f(B | A)$.
  Exercise: give example of greedy failing here.
sat, as tight polymatroidal elements

- We are given an $x \in P_f^+$ for submodular function $f$.
- Recall that for such an $x$, sat$(x)$ is defined as
  \[ \text{sat}(x) = \bigcup \{ A : x(A) = f(A) \} \]  \hspace{1cm} (10.22)
- We also have seen that sat$(x)$ can be defined as:
  \[ \text{sat}(x) = \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \]  \hspace{1cm} (10.23)
- We next show more formally that these are the same.

Let's start with one definition and derive the other.

\[ \text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \]  \hspace{1cm} (10.24)
\[ = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \]  \hspace{1cm} (10.25)
\[ = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\} \]  \hspace{1cm} (10.26)

This last bit follows since $1_e(A) = 1 \iff e \in A$. Continuing, we get
\[ \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\} \]  \hspace{1cm} (10.27)

- given that $x \in P_f^+$, meaning $x(A) \leq f(A)$ for all $A$, we must have
  \[ \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \right\} \]  \hspace{1cm} (10.28)
  \[ = \left\{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \right\} \]  \hspace{1cm} (10.29)
- So now, if $A$ is any set such that $x(A) = f(A)$, then we clearly have
  \[ \forall e \in A, e \in \text{sat}(x), \text{ and therefore that } \text{sat}(x) \supseteq A \]  \hspace{1cm} (10.30)
sat, as tight polymatroidal elements

- ...and therefore, with sat as defined in Eq. (10.14),
  \[
  \text{sat}(x) \supseteq \bigcup \{A : x(A) = f(A)\} \quad (10.31)
  \]

- On the other hand, for any \( e \in \text{sat}(x) \) defined as in Eq. (10.29), since \( e \) is itself a tight set, there is a set \( A \ni e \) such that \( x(A) = f(A) \), giving
  \[
  \text{sat}(x) \subseteq \bigcup \{A : x(A) = f(A)\} \quad (10.32)
  \]

- Therefore, the two definitions of sat are identical.

Saturation Capacity

- Another useful concept is saturation capacity which we develop next.
- For \( x \in P_f \), and \( e \in E \), consider finding
  \[
  \max \{\alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f\} \quad (10.33)
  \]

- This is identical to:
  \[
  \max \{\alpha : (x + \alpha 1_e)(A) \leq f(A), \forall A \ni \{e\}\} \quad (10.34)
  \]
  since any \( B \subseteq E \) such that \( e \notin B \) does not change in a \( 1_e \) adjustment, meaning \( (x + \alpha 1_e)(B) = x(B) \).
- Again, this is identical to:
  \[
  \max \{\alpha : x(A) + \alpha \leq f(A), \forall A \ni \{e\}\} \quad (10.35)
  \]
  or
  \[
  \max \{\alpha : \alpha \leq f(A) - x(A), \forall A \ni \{e\}\} \quad (10.36)
  \]
Saturation Capacity

- The max is achieved when
  \[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \] (10.37)
- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).
- Thus we have for \( x \in P_f \),
  \[ \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \] (10.38)
  \[ = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha e \in P_f \} \] (10.39)
- We immediately see that for \( e \in E \setminus \text{sat}(x) \), we have that \( \hat{c}(x; e) > 0 \).
- Also, for \( e \in \text{sat}(x) \), we have that \( \hat{c}(x; e) = 0 \).
- Note that any \( \alpha \) with \( 0 \leq \alpha \leq \hat{c}(x; e) \) we have \( x + \alpha e \in P_f \).
- We also see that computing \( \hat{c}(x; e) \) is a form of submodular function minimization.

Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given \( x \in P_f \), and \( e \in \text{sat}(x) \), define
  \[ D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \] (10.40)
  \[ = D(x) \cap \{ A : A \subseteq E, e \in A \} \] (10.41)
- Thus, \( D(x, e) \subseteq D(x) \), and \( D(x, e) \) is a sublattice of \( D(x) \).
- Therefore, we can define a unique minimal element of \( D(x, e) \) denoted as follows:
  \[ \text{dep}(x, e) = \begin{cases} \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \] (10.42)
- I.e., \( \text{dep}(x, e) \) is the minimal element in \( D(x) \) that contains \( e \) (the minimal \( x \)-tight set containing \( e \)).
The picture on the right summarizes the relationships between the lattices and sublattices.

Note, \( \bigcap_e \text{dep}(x, e) = \text{dep}(x) \).
An alternate expression for \( \text{dep} = \text{dry} \)

- Now, given \( x \in P_f \), and \( e \in \text{sat}(x) \), recall distributive sub-lattice of \( e \)-containing tight sets \( D(x, e) = \{ A : e \in A, x(A) = f(A) \} \).
- We can define the “1” element of this sub-lattice as:
  \[
  \text{sat}(x, e) \overset{\text{def}}{=} \bigcup \{ A : A \in D(x, e) \}.
  \]
- Analogously, we can define the “0” element of this sub-lattice as:
  \[
  \text{dry}(x, e) \overset{\text{def}}{=} \bigcap \{ A : A \in D(x, e) \}.
  \]
- We can see \( \text{dry}(x, e) \) as the elements that are necessary for \( e \)-containing tightness, with \( e \in \text{sat}(x) \).
- That is, we can view \( \text{dry}(x, e) \) as:
  \[
  \text{dry}(x, e) = \{ e' : x(A) < f(A), \forall A \not\ni e', e \in A \} \quad (10.44)
  \]
- This can be read as, for any \( e' \in \text{dry}(x, e) \), any \( e \)-containing set that does not contain \( e' \) is not tight for \( x \).
- But actually, \( \text{dry}(x, e) = \text{dep}(x, e) \), so we have derived another expression for \( \text{dep}(x, e) \) in Eq. (10.44).

Dependence Function and Fundamental Matroid Circuit

- Now, let \( (E, I) = (E, r) \) be a matroid, and let \( I \in I \) giving \( 1_I \in P_r \). Let \( e \in \text{sat}(1_I) = \text{span}(I) = \text{closure}(I) \).
- Given \( e \in \text{sat}(1_I) \setminus I \) and then consider an \( A \ni e \) with \( |I \cap A| = r(A) \).
- Then \( I \cap A \) serves as a base for \( A \) (i.e., \( I \cap A \) spans \( A \)) and any such \( A \) contains a circuit (i.e., we can add \( e \notin I \) to \( I \cap A \) w/o increasing rank).
- Given \( e \in \text{sat}(1_I) \setminus I \), and consider \( \text{dep}(1_I, e) \), with:
  \[
  \text{dep}(1_I, e) = \bigcap \{ A : e \in A \subseteq E, 1_I(A) = r(A) \} \quad (10.45)
  \]
  \[
  = \bigcap \{ A : e \in A \subseteq E, |I \cap A| = r(A) \} \quad (10.46)
  \]
- Then there is a unique minimal \( A \ni e \) with \( |I \cap A| = r(A) \).
- Thus, \( \text{dep}(1_I, e) \) must be a circuit since if it included more than a circuit, it would not be minimal in this sense.
Therefore, when \( e \in \text{sat}(1_I) \setminus I \), then \( \text{dep}(1_I, e) = C(I, e) \) where \( C(I, e) \) is the unique circuit contained in \( I + e \) in a matroid (the fundamental circuit of \( e \) and \( I \) that we encountered before).

Now, if \( e \in \text{sat}(1_I) \cap I \) with \( I \in I \), we said that \( C(I, e) \) was undefined (since no circuit is created in this case) and so we defined it as \( C(I, e) = \{e\} \).

In this case, for such an \( e \), we have \( \text{dep}(1_I, e) = \{e\} \) since all such sets \( A \ni e \) with \( |I \cap A| = r(A) \) contain \( e \), but in this case no cycle is created.

We are thus free to take subsets of \( I \) as \( A \), all of which must contain \( e \), but all of which have rank equal to size.

Also note: in general for \( x \in P_f \) and \( e \in \text{sat}(x) \), we have \( \text{dep}(x, e) \) is tight by definition.

Summary of \( \text{sat} \), and \( \text{dep} \)

- For \( x \in P_f \), \( \text{sat}(x) \) (span, closure) is the maximal saturated (\( x \)-tight) set w.r.t. \( x \). I.e., \( \text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha e \notin P_f\} \). That is,

\[
\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} = \bigcup \{A : A \subseteq E, x(A) = f(A)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha e \notin P_f\}
\]

- For \( e \in \text{sat}(x) \), \( \text{dep}(x, e) \) (fundamental circuit) is the minimal (common) saturated (\( x \)-tight) set w.r.t. \( x \) containing \( e \). That is,

\[
\text{dep}(x, e) = \begin{cases} 
\bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\
\emptyset & \text{else}
\end{cases} = \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1_e - 1_{e'}) \in P_f\}
\]
Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin I$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha 1_e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
- Recall, we have $C(I, e) \setminus e' \in I$ for $e' \in C(I, e)$. i.e., $C(I, e)$ consists of elements that when removed recover independence.
- In other words, for $e \in \text{span}(I) \setminus I$, we have that
  \[ C(I, e) = \{ a \in E : I + e - a \in I \} \quad (10.51) \]
  i.e., an addition of $e$ to $I$ stays within $I$ only if we simultaneously remove one of the elements of $C(I, e)$.
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x, e)$ property to take the form: a positive move in the $e$-direction stays within $P_f^+$ only if we simultaneously take a negative move in one of the $\text{dep}(x, e)$ directions.

Dependence Function and exchange in 2D

- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:
  \[ (e) \]

  Left: $A \cap \text{dep}(x, e) = \emptyset$, and we can't move further in $(e)$ direction, and moving in any negative $a \in A$ direction doesn't change that. Notice no dependence between $(e)$ and any element in $A$.

  Right: $A \subseteq \text{dep}(x, e)$, and we can't move further in the $(e)$ direction, but we can move further in $(e)$ direction by moving in some $a \in A$ negative direction. Notice dependence between $(e)$ and elements in $A$. 

We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.

In 3D, we have:

\[ \text{(e)-(a)} \]

I.e., for \( e \in \text{sat}(x) \), \( a \in \text{sat}(x) \), \( a \in \text{dep}(x, e) \), \( e \notin \text{dep}(x, a) \), and
\[
\text{dep}(x, e) = \{ a : a \in E, \exists \alpha > 0 : x + \alpha(1_e - 1_a) \in P_f \} \quad (10.52)
\]

We next show this formally ...

The derivation for \( \text{dep}(x, e) \) involves turning a strict inequality into a non-strict one with a strict explicit slack variable \( \alpha \):

\[
\text{dep}(x, e) = \{ e' : x(A) < f(A), \forall A \nsubseteq e', e \in A \} \quad (10.53)
\]

\[
= \{ e' : \exists \alpha > 0, \ \text{s.t.} \ \alpha \leq f(A) - x(A), \forall A \nsubseteq e', e \in A \} \quad (10.54)
\]

\[
= \{ e' : \exists \alpha > 0, \ \text{s.t.} \ \alpha 1_e(A) \leq f(A) - x(A), \forall A \nsubseteq e', e \in A \} \quad (10.55)
\]

\[
= \{ e' : \exists \alpha > 0, \ \text{s.t.} \ \alpha(1_e(A) - 1_{e'}(A)) \leq f(A) - x(A), \forall A \nsubseteq e', e \in A \} \quad (10.56)
\]

\[
= \{ e' : \exists \alpha > 0, \ \text{s.t.} \ x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A \nsubseteq e', e \in A \} \quad (10.57)
\]

Now, \( 1_e(A) - 1_{e'}(A) = 0 \) if either \( \{ e, e' \} \subseteq A \), or \( \{ e, e' \} \cap A = \emptyset \).

Also, if \( e' \in A \) but \( e \notin A \), then
\[
x(A) + \alpha(1_e(A) - 1_{e'}(A)) = x(A) - \alpha \leq f(A) \quad \text{since} \ x \in P_f.
\]
dep and exchange derived

- thus, we get the same in the above if we remove the constraint \( A \not\ni e', e \in A \), that is we get

\[
\text{dep}(x,e) = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A\}
\]

(10.59)

- This is then identical to

\[
\text{dep}(x,e) = \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(1_e - 1_{e'}) \in P_f\}
\]

(10.60)

- Compare with original, the minimal element of \( D(x,e) \), with \( e \in \text{sat}(x) \):

\[
\text{dep}(x,e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\
\emptyset & \text{else} \end{cases}
\]

(10.61)

Sources for Today’s Lecture

End

Scratch Paper