Announcements, Assignments, and Reminders

- Reminder: class web page is at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)

- Please do use our discussion board (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.
Outstanding Reading

- Read chapter 1 from Fujishige book.
- Read over lecture slides, all posted on our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/).
- See the summary slide at the end for some additional ideas for reading. A good summary of matroid properties is http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
Submodular Definitions

**Definition 4.2.4 (submodular concave)**

A function \( f : 2^V \rightarrow \mathbb{R} \) is submodular if for any \( A, B \subseteq V \), we have that:

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B)
\]  

(4.7)

An alternate and (as we see in lecture 3) equivalent definition is:

**Definition 4.2.5 (diminishing returns)**

A function \( f : 2^V \rightarrow \mathbb{R} \) is submodular if for any \( A \subseteq B \subset V \), and \( v \in V \setminus B \), we have that:

\[
f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)
\]  

(4.8)

This means that the incremental “value”, “gain”, or “cost” of \( v \) decreases (diminishes) as the context in which \( v \) is considered grows from \( A \) to \( B \).
An alternate and equivalent definition is:

Definition 4.2.6 (group diminishing returns)

A function $f : 2^V \rightarrow \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $C \subseteq V \setminus B$, we have that:

$$f(A \cup C) - f(A) \geq f(B \cup C) - f(B) \quad (4.24)$$

This means that the incremental “value” or “gain” of set $C$ decreases as the context in which $v$ is considered grows from $A$ to $B$ (diminishing returns)
We want to show that **Submodular Concave** (Definition 4.2.4), **Diminishing Returns** (Definition 4.2.5), and **Group Diminishing Returns** (Definition 4.2.6) are identical. We will show that:

- Submodular Concave $\Rightarrow$ Diminishing Returns
- Diminishing Returns $\Rightarrow$ Group Diminishing Returns
- Group Diminishing Returns $\Rightarrow$ Submodular Concave
Proof.

- Assume Submodular concave, so \( \forall A, B \) we have
  \[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \]

- Given \( A, B \) and \( v \in V \) such that: \( A \subseteq B \subseteq V \setminus \{v\} \), we have from submodular concave that:
  \[ f(A + v) + f(B) \geq f(B + v) + f(A) \quad (4.24) \]

- Rearranging, we have
  \[ f(A + v) - f(A) \geq f(B + v) - f(B) \quad (4.25) \]
Proof.

Let $C = \{c_1, c_2, \ldots, c_k\}$. Then diminishing returns implies

$$f(A \cup C) - f(A)$$

$$= f(A \cup C) - \sum_{i=1}^{k-1} \left( f(A \cup \{c_1, \ldots, c_i\}) - f(A \cup \{c_1, \ldots, c_i\}) \right) - f(A) \quad (4.24)$$

$$= \sum_{i=1}^{k} f(A \cup \{c_1 \ldots c_i\}) - f(A \cup \{c_1 \ldots c_{i-1}\}) \quad (4.25)$$

$$\geq \sum_{i=1}^{k} f(B \cup \{c_1 \ldots c_i\}) - f(B \cup \{c_1 \ldots c_{i-1}\}) \quad (4.26)$$

$$= f(B \cup C) - \sum_{i=1}^{k-1} \left( f(B \cup \{c_1, \ldots, c_i\}) - f(B \cup \{c_1, \ldots, c_i\}) \right) - f(B) \quad (4.27)$$

$$= f(B \cup C) - f(B) \quad (4.28)$$

$$= f(B \cup C) - f(B) \quad (4.29)$$
Proof.

Assume group diminishing returns. Assume \( A \neq B \) otherwise trivial. Define \( A' = A \cap B \), \( C = A \setminus B \), and \( B' = B \). Then

\[
f(A' + C) - f(A') \geq f(B' + C) - f(B') \tag{4.24}
\]
giving

\[
f(A' + C) + f(B') \geq f(B' + C) + f(A') \tag{4.25}
\]
or

\[
f(A \cap B + A \setminus B) + f(B) \geq f(B + A \setminus B) + f(A \cap B) \tag{4.26}
\]

which is the same as the submodular concave condition

\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{4.27}
\]
Many (Equivalent) Definitions of Submodularity

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq E \quad (4.30) \]

\[ f(j|S) \geq f(j|T), \quad \forall S \subseteq T \subseteq E, \text{ with } j \in E \setminus T \quad (4.31) \]

\[ f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq E, \text{ with } C \subseteq E \setminus T \quad (4.32) \]

\[ f(j|S) \geq f(j|S \cup \{k\}), \forall S \subseteq E \text{ with } j \in E \setminus (S \cup \{k\}) \quad (4.33) \]

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \forall S, T \subseteq E \quad (4.34) \]

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \forall S \subseteq T \subseteq E \quad (4.35) \]

\[ f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \forall S, T \subseteq E \quad (4.36) \]

\[ f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \forall T \subseteq S \subseteq E \quad (4.37) \]
Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an independent set.

**Definition 4.2.8 (Matroid)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

- (I1) \( \emptyset \in \mathcal{I} \)
- (I2) \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \)
- (I3) \( \forall I, J \in \mathcal{I}, \text{ with } |I| = |J| + 1, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I} \).
Matroid

Slight modification (non unit increment) that is equivalent.

**Definition 4.2.8 (Matroid-II)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

(I1') \(\emptyset \in \mathcal{I}\)

(I2') \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (or “down-closed”)

(I3') \(\forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}\)

Note (I1)≡(I1'), (I2)≡(I2'), and we get (I3)≡(I3') using induction.
Why do we call the $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ definition of submodularity, submodular concave?
Why do we call the \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) definition of submodularity, submodular concave?

A continuous twice differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).
Submodular Concave

- Why do we call the \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) definition of submodularity, submodular concave?
- A continuous twice differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).
- Define a “discrete derivative” or difference operator defined on discrete functions \( f : 2^V \rightarrow \mathbb{R} \) as follows:
  \[
  (\nabla_B f)(A) \triangleq f(A \cup B) - f(A \setminus B) = f(B|A \setminus B)
  \]
  read as: the derivative of \( f \) at \( A \) in the direction \( B \).
Submodular Concave

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- A continuous twice differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).
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\[
(\nabla_B f)(A) \triangleq f(A \cup B) - f(A \setminus B) = f(B | (A \setminus B)) \tag{4.1}
\]

read as: the derivative of \( f \) at \( A \) in the direction \( B \).
- Hence, if \( A \cap B = \emptyset \), then \( (\nabla_B f)(A) = f(B | A) \).
Submodular Concave

- Why do we call the \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) definition of submodularity, submodular concave?
- A continuous twice differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).
- Define a “discrete derivative” or difference operator defined on discrete functions \( f : 2^V \to \mathbb{R} \) as follows:
  \[
  (\nabla_B f)(A) \overset{\Delta}{=} f(A \cup B) - f(A \setminus B) = f(B|A \setminus B) \tag{4.1}
  \]
  read as: the derivative of \( f \) at \( A \) in the direction \( B \).
- Hence, if \( A \cap B = \emptyset \), then \( (\nabla_B f)(A) = f(B|A) \).
- Consider a form of second derivative or 2nd difference:
  \[
  (\nabla_C \nabla_B f)(A) = \nabla_C [f(A \cup B) - f(A \setminus B)]
  = f(A \cup B \cup C) - f((A \cup C) \setminus B)
  \]
  \[
  - f((A \setminus C) \cup B) + f((A \setminus C) \setminus B) \tag{4.2}
  \]
Submodular Concave

- If the second difference operator everywhere nonpositive:

\[
\begin{align*}
    f(A \cup B \cup C) - f((A \cup C) \setminus B) - \\
    f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0
\end{align*}
\] (4.3)
If the second difference operator everywhere nonpositive:

\[
f(A \cup B \cup C) - f((A \cup C) \setminus B) \\
- f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0
\]  

(4.3)

then we have the equation:

\[
f((A \cup C) \setminus B) + f((A \setminus C) \cup B) \geq f(A \cup B \cup C) + f(A \setminus C \setminus B)
\]  

(4.4)
If the second difference operator everywhere nonpositive:

\[ f(A \cup B \cup C) - f((A \cup C) \setminus B) \]
\[ - f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0 \]  

(4.3)

then we have the equation:

\[ f((A \cup C) \setminus B) + f((A \setminus C) \cup B) \geq f(A \cup B \cup C) + f(A \setminus C \setminus B) \]  

(4.4)

Define \( A' = (A \cup C) \setminus B \) and \( B' = (A \setminus C) \cup B \). Then the above implies:

\[ f(A') + f(B') \geq f(A' \cup B') + f(A' \cap B') \]  

(4.5)

and note that \( A' \) and \( B' \) so defined can be arbitrary.
Submodular Concave

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\[
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- f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0
\]  

(4.3)

then we have the equation:

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 f((A \cup C) \setminus B) + f((A \setminus C) \cup B) \geq f(A \cup B \cup C) + f(A \setminus C \setminus B)
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 f(A') + f(B') \geq f(A' \cup B') + f(A' \cap B')
\]  

(4.5)

and note that \( A' \) and \( B' \) so defined can be arbitrary.

- One sense in which submodular functions are like concave functions.
Submodular Concave

(a) \( A' = (A \cup C) \setminus B \)

(b) \( B' = (A \setminus C) \cup B \)

Figure: A figure showing \( A' \cup B' = A \cup B \cup C \) and \( A' \cap B' = A \setminus C \setminus B \).
(c) \( A' = (A \cup C) \setminus B \)

(d) \( B' = (A \setminus C) \cup B \)

Figure: A figure showing \( A' \cup B' = A \cup B \cup C \) and \( A' \cap B' = A \setminus C \setminus B \).
Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise $A$ is called dependent.
Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called \textit{independent} if $A \in \mathcal{I}$ and otherwise $A$ is called \textit{dependent}.

For $U \subseteq E$, a subset $B \subseteq U$ is called a \textit{base} of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$. 
Matroids

- Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise $A$ is called dependent.

- For $U \subseteq E$, a subset $B \subseteq U$ is called a base of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

- If $U = E$, then a “base of $E$” is just called a base of the matroid $M$ (this corresponds to a basis in a linear space).
Matroids - important property

**Proposition 4.4.1**

*In a matroid $M = (E, I)$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.*
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In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.

- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
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- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show this.
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- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show this
- The common size of all the bases of $U$ is called the rank of $U$, denoted $r_M(U)$ or just $r(U)$ when the matroid in equation is unambiguous.
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- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show this.
- The common size of all the bases of $U$ is called the rank of $U$, denoted $r_M(U)$ or just $r(U)$ when the matroid in equation is unambiguous.
- $r(E) = r(E,I)$ is the rank of the matroid, and is the common size of all the bases of the matroid.
We can a bit more formally define the rank function this way.

**Definition 4.4.2**

The rank of a matroid is a function \( r : 2^E \rightarrow \mathbb{Z}_+ \) defined by

\[
r(A) = \max \{ |X| : X \subseteq A, X \in \mathcal{I} \}
\] 

(4.6)
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$$r(A) = \max \{ |X| : X \subseteq A, X \in \mathcal{I} \}$$

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- From the above, we immediately see that $r(A) \leq |A|$. 
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r(A) = \max \{|X| : X \subseteq A, X \in \mathcal{I}\}
\]  

(4.6)

- From the above, we immediately see that \( r(A) \leq |A| \).
- Moreover, if \( r(A) = |A| \), then \( A \in \mathcal{I} \), meaning \( A \) is independent (in this case, \( A \) is a self base).
Lemma 4.4.3

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]
Lemma 4.4.3

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is \( r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \).

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
Lemma 4.4.3

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \). (We can find such a \( Y \supseteq X \) because, starting from \( X \subseteq A \cup B \), and since \(|Y| \geq |X|\), we can choose a \( y \in Y \subseteq A \cup B \) such that \( X + y \in \mathcal{I} \) but since \( y \in A \cup B \), also \( X + y \in A \cup B \). We can keep doing this while \(|Y| > |X|\) since this is a matroid.)
Lemma 4.4.3

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$.
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
Lemma 4.4.3

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is
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r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
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Proof.

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3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have
\[
r(A) + r(B) \geq |X| + |Y| = r(A \cap B) + r(A \cup B)
\]
Lemma 4.4.3

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
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Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \)
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).
3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have
\[
r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \quad (4.7)
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Lemma 4.4.3

The rank function $r : 2^E \rightarrow \mathbb{Z}_+$ of a matroid is submodular, that is
$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$.
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B|$$

$$\quad = |Y \cap (A \cap B)| + |Y \cap (A \cup B)|$$

(4.7)

(4.8)
Matroids - rank

Lemma 4.4.3

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
\text{rank}(A) + \text{rank}(B) \geq \text{rank}(A \cup B) + \text{rank}(A \cap B)
\]

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).
3. Since \( M \) is a matroid, we know that \( \text{rank}(A \cap B) = \text{rank}(X) = |X| \), and \( \text{rank}(A \cup B) = \text{rank}(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( \text{rank}(A) \geq |A \cap U| \).
4. Then we have
\[
\text{rank}(A) + \text{rank}(B) \geq |Y \cap A| + |Y \cap B|
\]
\[
= |Y \cap (A \cap B)| + |Y \cap (A \cup B)|
\]
\[
\geq |X| + |Y| = \text{rank}(A \cap B) + \text{rank}(A \cup B)
\]
In fact, we can use the rank of a matroid for its definition.

**Theorem 4.4.4 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
(R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- A matroid is sometimes given as $(E, r)$ where $E$ is ground set and $r$ is rank function.
Matroids

In fact, we can use the rank of a matroid for its definition.

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Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \quad 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)

(R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$. 

Prof. Jeff Bilmes
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## Matroids from rank

**Proof of Theorem 4.4.4 (matroid from rank).**

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 4.6 satisfies (R1), (R2), and, as we saw in Lemma 4.4.3, (R3) too.
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- Next, assume we have (R1), (R2), and (R3). Define \( \mathcal{I} = \{ X \subseteq E : r(X) = |X| \} \). We will show that \((E, \mathcal{I})\) is a matroid.
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$$r(X) \geq r(Y) - r(Y \setminus X) \quad (4.10)$$
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    r(X) &\geq r(Y) - r(Y \setminus X) - r(\emptyset) \\
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- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \quad (4.10)$$

$$\geq |Y| - |Y \setminus X| \quad (4.11)$$

$$= |X| \quad (4.12)$$
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  \[ \mathcal{I} = \{ X \subseteq E : r(X) = |X| \} \]. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,
  \[
  r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \\
  \geq |Y| - |Y \setminus X| \\
  = |X|
  \]

  implying $r(X) = |X|$, and thus $X \in \mathcal{I}$. 

Proof of Theorem 4.4.4 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$. 

Proof of Theorem 4.4.4 (matroid from rank) cont.

- Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \).

- Suppose, to the contrary, that \( \forall b \in B \setminus A, r(A + b) \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| \). Then
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$$r(B) \leq r(A \cup B) \quad (4.13)$$
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  \[
  r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)
  \]  
  (4.13)  
  (4.14)
Proof of Theorem 4.4.4 (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \).

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\begin{align*}
    r(B) & \leq r(A \cup B) \quad (4.13) \\
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    & = r(A \cup (B \setminus \{b_1\})) \quad (4.15)
\end{align*}
\]
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$.

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giving a contradiction since $B \in \mathcal{I}$.  

Proof of Theorem 4.4.4 (matroid from rank) cont.
Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \).

Suppose, to the contrary, that \( \forall b \in B \setminus A, r(A + b) \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| \). Then

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\begin{align*}
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\end{align*}
\]
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$.

Suppose, to the contrary, that $\forall b \in B \setminus A$, $r(A + b) \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\begin{align*}
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    & = r(A \cup (B \setminus \{b_1\})) \quad (4.14) \\
    & \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \quad (4.15) \\
    & = r(A \cup (B \setminus \{b_1, b_2\})) \quad (4.16) \\
    & \leq \ldots \leq r(A) = |A| < |B| \quad (4.17) \\

giving a contradiction since $B \in \mathcal{I}$. 
\end{align*}
Another way of using function $r$ to define a matroid.

**Theorem 4.4.5 (Matroid from rank II)**

Let $E$ be a finite set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$. 
Definition 4.4.6 (closed/flat/subspace)

A subset $A \subseteq E$ is closed or a flat or a subspace of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$. 
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Definition 4.4.7 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$. 
Matroids, other definitions using matroid rank $r : 2^V \rightarrow \mathbb{Z}_+$

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Therefore, a closed set $A$ has $\text{span}(A) = A$.

**Definition 4.4.8 (circuit)**

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.
Theorem 4.4.9 (Matroid (by bases))

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
Matroids by bases

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Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 4.4.10 (Matroid by circuits)**

Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.

1. $C$ is the collection of circuits of a matroid;
2. if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;
Matroids by circuits

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Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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- Independence (define the independent sets).
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this yesterday, but it is possible)
Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that
  $c(X) = \sum_{x \in X} c(x)$ is maximum.

- This seems remarkably similar to the max spanning tree problem.
Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.

- This sounds like a set cover problem (find the minimum cost covering set of sets).
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{ A \subseteq E : |A| \leq k \}$. 

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Rank function $r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases}$

Therefore, this function is submodular. Not surprising since $r(A) = \min(|A|, k)$ which is a non-decreasing concave function applied to a modular function.

Closure function $\text{span}(A) = \begin{cases} A & \text{if } |A| < k \\ E & \text{if } |A| \geq k \end{cases}$

A "free" matroid sets $k = |E|$, so everything is independent.
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- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
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- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \not\in I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$. 
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$$r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases} \quad (4.19)$$
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- A “free” matroid sets $k = |E|$, so everything is independent.
Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
Linear (or Matric) Matroid

- Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
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- the rank function is just the rank of the space spanned by the corresponding set of vectors.
Linear (or Matric) Matroid

- Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
- The rank function is just the rank of the space spanned by the corresponding set of vectors.
- Rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
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- Called both linear matroids and matric matroids.
Let $G = (V, E)$ be a graph. Consider $(E, I)$ where the edges of the graph $E$ are the ground set and $A \in I$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.
Cycle Matroid of a graph: Graphic Matroids

Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, \mathcal{I})$ is a matroid.
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Bases are spanning forests (spanning trees if $G$ is connected).
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Closure function adds all edges between the vertices adjacent to any edge in $A$. Closure of a spanning forest is $G$. 

Partition Matroid

- Let $V$ be our ground set.
Partition Matroid

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- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

\[ I = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \quad (4.21) \]

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.
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- Note that a \( k \)-uniform matroid is a trivial example of a partition matroid with \( \ell = 1, V_1 = V \), and \( k_1 = k \).

- We’ll show that property (I3’) in Def 4.2.8 holds. If \( X, Y \in \mathcal{I} \) with \( |Y| > |X| \), then there must be at least one \( i \) with \( |Y \cap V_i| > |X \cap V_i| \). Therefore, adding one element \( e \in V_i \cap (Y \setminus X) \) to \( X \) won’t break independence.
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$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)$$

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2. min(submodular($A$), $k_i$) is submodular in $A$ since $|A \cap V_i|$ is monotone.
3. Sums of submodular functions are submodular.

- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
A partition matroid can be viewed using a bipartite graph.

Letting $V$ denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where $V$ is the ground set, $I$ is a set of “indices”, and $E$ is the set of edges.

$I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into $\ell$ clusters, where there are $k_i$ nodes in the $i^{th}$ group $I_i$.

$(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$. 
Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$
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Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
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For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) = \text{maximum matching involving } X$. 

\begin{itemize}
  \item \text{Partition Matroid, rank as matching}
  \item \text{Example where } \ell = 5, \quad (k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).
  \item Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
  \item Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
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Laminar Matroid

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A set system $(V, \mathcal{F})$ is called a laminar family of for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \setminus B$, $B \setminus A$, or $A \cap B$ is empty.
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- Family is laminar if it has no two “properly intersecting” members: i.e., intersecting \(A \cap B \neq \emptyset\) and not comparable (one is not contained in the other).
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Suppose we have a laminar family \(F\) of subsets of \(V\) and an integer \(k(A)\) for every set \(A \in F\).
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Suppose we have a laminar family $\mathcal{F}$ of subsets of $V$ and an integer $k(A)$ for every set $A \in \mathcal{F}$.

Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{ I \subseteq E : |X \cap A| \leq k(A) \text{ for all } A \in \mathcal{F} \} \quad (4.23)$$
System of Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(\emptyset \subset V_i \subseteq V\) for all \(i\)).
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A family \((v_i : i \in I)\) with \(v_i \in V\) for index set \(I\) is said to be a system of representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \to I\) such that \(v_i \in V_{\pi(i)}\). \(v_i\) is the representative of set \(\pi(i)\), meaning the \(i^{th}\) representative is meant to represent set \(V_{\pi(i)}\). Consider the house of representatives, \(v_i = \text{“John Smith”}\), while \(i = \text{King County}\).
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- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have \(v_1 \in T\), where \(v_1\) represents both \(V_1\) and \(V_2\).
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- We can view this as a bipartite graph.
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- Here, $\ell = 6$, and $V = (V_1, V_2, \ldots, V_6)$
  
  \[ V = (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}) \]
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- The representatives are shown as colors on the left.
- Here, the set of representatives is not distinct. In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
System of Distinct Representatives

Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $V_i \subseteq V$ for all $i$).
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- A family \((v_i : i \in I)\) with \(v_i \in V\) for index set \(I\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
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**Definition 4.7.1 (transversal)**

Given a set system \((V, \mathcal{V})\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[
x \in V_{\pi(x)} \text{ for all } x \in T
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- Note that due to it being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct).