Announcements, Assignments, and Reminders

- Reminder: class web page is at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)

- Please do use our discussion board (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.
Read chapter 1 from Fujishige book.

Read over lecture slides, all posted on our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/).

See the summary slide at the end for some additional ideas for reading. A good summary of matroid properties is http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
Submodular Concave

- Why do we call the \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) definition of submodularity, submodular \textit{concave}?

- A continuous twice differentiable function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).

- Define a “discrete derivative” or difference operator defined on discrete functions \( f : 2^V \rightarrow \mathbb{R} \) as follows:

\[
(\nabla_B f)(A) \triangleq f(A \cup B) - f(A \setminus B) = f(B | (A \setminus B)) \quad (5.1)
\]

read as: the derivative of \( f \) at \( A \) in the direction \( B \).

- Hence, if \( A \cap B = \emptyset \), then \( (\nabla_B f)(A) = f(B | A) \).

- Consider a form of second derivative or 2nd difference:

\[
(\nabla_C \nabla_B f)(A) = \nabla_C [f(A \cup B) - f(A \setminus B)] \\
= f(A \cup B \cup C) - f((A \cup C) \setminus B) \\
- f((A \setminus C) \cup B) + f((A \setminus C) \setminus B) \quad (5.2)
\]
Submodular Concave

- If the second difference operator everywhere nonpositive:

\[
f(A \cup B \cup C) - f((A \cup C) \setminus B) - f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0 \tag{5.1}
\]

then we have the equation:

\[
f((A \cup C) \setminus B) + f((A \setminus C) \cup B) \geq f(A \cup B \cup C) + f(A \setminus C \setminus B) \tag{5.2}
\]

- Define \( A' = (A \cup C) \setminus B \) and \( B' = (A \setminus C) \cup B \). Then the above implies:

\[
f(A') + f(B') \geq f(A' \cup B') + f(A' \cap B') \tag{5.3}
\]

and note that \( A' \) and \( B' \) so defined can be arbitrary.

- One sense in which submodular functions are like concave functions.
Submodular Concave

Figure: A figure showing $A' \cup B' = A \cup B \cup C$ and $A' \cap B' = A \setminus C \setminus B$. 

(a) $A' = (A \cup C) \setminus B$

(b) $B' = (A \setminus C) \cup B$
(c) \( A' = (A \cup C) \setminus B \)

(d) \( B' = (A \setminus C) \cup B \)

Figure: A figure showing \( A' \cup B' = A \cup B \cup C \) and \( A' \cap B' = A \setminus C \setminus B \).
Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an independent set.

**Definition 5.2.4 (Matroid)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

1. \( \emptyset \in \mathcal{I} \)
2. \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \)
3. \( \forall I, J \in \mathcal{I}, \text{ with } |I| = |J| + 1, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I} \).
Matroid

Slight modification (non unit increment) that is equivalent.

**Definition 5.2.4 (Matroid-II)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

1. \((I1')\) \(\emptyset \in \mathcal{I}\)
2. \((I2')\) \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (or “down-closed”)
3. \((I3')\) \(\forall I, J \in \mathcal{I}, \text{with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}\)

Note \((I1)\equiv(I1'), (I2)\equiv(I2'), \text{ and we get } (I3)\equiv(I3') \text{ using induction.}\)
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this yesterday, but it is possible)
Matroids, other definitions using matroid rank \( r : 2^V \to \mathbb{Z}_+ \)

**Definition 5.2.9 (closed/flat/subspace)**

A subset \( A \subseteq E \) is **closed** or a **flat** or a **subspace** of matroid \( M \) if for all \( x \in E \setminus A \), \( r(A \cup \{x\}) = r(A) + 1 \).

**Definition 5.2.10 (closure)**

Given \( A \subseteq E \), the **closure** (or **span**) of \( A \), is defined by

\[
\text{span}(A) = \{ b \in E : r(A \cup \{b\}) = r(A) \}.
\]

Therefore, a closed set \( A \) has \( \text{span}(A) = A \).

**Definition 5.2.11 (circuit)**

A subset \( A \subseteq E \) is **circuit** or a **cycle** if it is an inclusionwise minimally dependent set (i.e., if \( r(A) < |A| \) and for any \( a \in A \), \( r(A \setminus \{a\}) = |A| - 1 \)).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.
Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

$$I = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \quad (5.16)$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.

- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

- We’ll show that property (I3’) in Def ?? holds. If $X, Y \in I$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
Partition Matroid

- What is the partition matroid’s rank function?
- A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  \hspace{1cm} (5.16)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (even modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
3. Sums of submodular functions are submodular.

- \(r(A)\) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
System of Representatives

- We can view this as a bipartite graph. The groups of $V$ are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$, and $\mathcal{V} = (V_1, V_2, \ldots, V_6)$
  $$= (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}).$$

\[ V \begin{array}{c}
\begin{array}{cccccccc}
a & b & c & d & e & f & g & h \\
6 & 5 & 4 & 3 & 2 & 1 & & \\
\end{array}
\end{array} I \]
System of Representatives

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- A system of representatives would make sure that there is a representative for each color group. For example,
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- The representatives are shown as colors on the left.
System of Representatives

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- Here, $\ell = 6$, and $\mathcal{V} = (V_1, V_2, \ldots, V_6) = (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\})$.

A system of representatives would make sure that there is a representative for each color group. For example,

- The representatives are shown as colors on the left.
- Here, the set of representatives is not distinct. In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
System of Distinct Representatives

Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)). Hence, \(|I| = |\mathcal{V}|\).

A family \((v_i : i \in I)\) with \(v_i \in V\) for index set \(I\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).

In a system of distinct representatives, there is a requirement for the representatives to be distinct. Let's re-state (and rename) this as a:

**Definition 5.2.9 (transversal)**

Given a set system \((V, \mathcal{V})\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[x \in V_{\pi(x)} \text{ for all } x \in T\]  \hspace{1cm} (5.17)

Note that due to it being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct).
A set $X \subseteq V$ is a **partial transversal** if $X$ is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$. 
A set $X \subseteq V$ is a **partial transversal** if $X$ is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.

Therefore, for any transversal $T$, any subset $T' \subseteq T$ is a partial transversal (down closed).
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[
V(J) = \bigcup_{j \in J} V_j
\]

so \(|V(J)|\) is the set cover function.
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V(J) = \bigcup_{j \in J} V_j
\]  

(5.1)

so \(|V(J)|\) is the set cover function.

- We have

**Theorem 5.3.1 (Hall’s theorem)**

Given a set system \((V, \mathcal{V})\), the family of subsets \(\mathcal{V} = (V_i : i \in I)\) has a transversal \((v_i : i \in I)\) iff for all \(J \subseteq I\)

\[
|V(J)| \geq |J|
\]  

(5.2)
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

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- Hall’s theorem as a bipartite graph.
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- Hall’s theorem as a bipartite graph.
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?

- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[
V(J) = \bigcup_{j \in J} V_j
\]

so \(|V(J)|\) is the set cover function.

Moreover, we have

**Theorem 5.3.2 (Rado's theorem)**

*If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) which is independent in \(M\) iff for all \(J \subseteq I\)*

\[
r(V(J)) \geq |J|
\]
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

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V(J) = \bigcup_{j \in J} V_j
\]

so \(|V(J)|\) is the set cover function.
- Moreover, we have

**Theorem 5.3.2 (Rado’s theorem)**

*If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) which is independent in \(M\) iff for all \(J \subseteq I\)*

\[
r(V(J)) \geq |J|
\]

- Note, a transversal \(T\) independent in \(M\) means that \(r(T) = |T|\).
More general conditions for existence of transversals

**Theorem 5.3.3**

If $\mathcal{V} = (V_i : I \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\bigcup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I$$  \hspace{1cm} (5.4)

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I$$  \hspace{1cm} (5.5)
More general conditions for existence of transversals

**Theorem 5.3.3**

If $\mathcal{V} = (V_i : I \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \rightarrow \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

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(5.4)

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I$$

(5.5)

Given Theorem 5.3.3, we immediately get Theorem 5.3.1 by taking $f(S) = |S|$ for $S \subseteq V$. In which case, Eq. 5.4 requires the system of representatives to be distinct.
More general conditions for existence of transversals

**Theorem 5.3.3**

If $\mathcal{V} = (V_i : I \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

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(5.4)

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(5.5)

- Given Theorem 5.3.3, we immediately get Theorem 5.3.1 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 5.3.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. where, Eq. 5.4 insists the system of representatives is independent in $M$. 

Prof. Jeff Bilmes
EE596A/Fall 2012/Submodularity – Lecture 5 - October 10th, 2012
More general conditions for existence of transversals

first part proof of Theorem 5.3.3.

- Suppose Eq. 5.4 is true. Then since $f$ is monotone, and since $V(J) \supseteq \bigcup_{i \in J} \{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 5.5 immediately follows.

...
More general conditions for existence of transversals

Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

When Eq. 5.5 and the above holds, this means that for any subsets \( J_1, J_2 \subseteq I \setminus \{1\} \), we have that

\[
\begin{align*}
  f(V_1 \cup V(J_1)) & \geq |J_1| + 1 \quad (5.6) \\
  f(V_1 \cup V(J_2)) & \geq |J_2| + 1 \quad (5.7)
\end{align*}
\]
More general conditions for existence of transversals

Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take \( \bar{v}_1, \bar{v}_2 \in V_1 \) as two distinct elements in \( V_1 \),
More general conditions for existence of transversals

Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take \( \bar{v}_1, \bar{v}_2 \in V_1 \) as two distinct elements in \( V_1 \),

- and there must exist subsets \( J_1, J_2 \) of \( I \setminus \{1\} \) such that

\begin{align*}
    f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) &< |J_1| + 1, \quad (5.8) \\
    f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) &< |J_2| + 1, \quad (5.9)
\end{align*}

(note that either one or both of \( J_1, J_2 \) could be empty).
More general conditions for existence of transversals

Lemma 5.3.4

Suppose Eq. 5.5 \((f(V(J)) \geq |J|, \forall J \subseteq I)\) is true for \(V\), and there exists an \(i\) such that \(|V_i| \geq 2\) (w.l.o.g., say \(i = 1\)). Then there exists \(\bar{v} \in V_1\) such that the family of subsets \((V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n)\) also satisfies Eq 5.5.

Proof.

Taking \(X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)\) and \(Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)\), we have \(f(X) \leq |J_1|\), \(f(Y) \leq |J_2|\), and that:

\[
X \cup Y = V_1 \cup V(J_1 \cup J_2),
\]

(5.10)

\[
X \cap Y \supseteq V(J_1 \cap J_2),
\]

(5.11)

and

\[
|J_1| + |J_2| \geq f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).
\]

(5.12)
More general conditions for existence of transversals

Lemma 5.3.4

Suppose Eq. 5.5 \( (f(V(J)) \geq |J|, \forall J \subseteq I) \) is true for \( \mathcal{V} \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \((V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n)\) also satisfies Eq 5.5.

Proof.

- since \( f \) submodular monotone non-decreasing, & Eqs 5.6-5.9,

\[
|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2))
\]  

(5.13)
More general conditions for existence of transversals

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Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

- since \( f \) submodular monotone non-decreasing, & Eqs 5.6-5.9,

\[
|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{5.13}
\]

- Since \( V \) satisfies Eq. 5.5, \( 1 \notin J_1 \cup J_2 \), & Eqs 5.6-5.7, this gives

\[
|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{5.14}
\]

which is a contradiction since cardinality is modular.
More general conditions for existence of transversals

Converse proof of Theorem 5.3.3.

Conversely, suppose Eq. 5.5 is true.
More general conditions for existence of transversals

Converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
More general conditions for existence of transversals

converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ until we arrive at a family of singleton sets.
converse proof of Theorem 5.3.3.

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- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ until we arrive at a family of singleton sets.
- This family will be the required system of representatives.
More general conditions for existence of transversals

Converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ until we arrive at a family of singleton sets.
- This family will be the required system of representatives.

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.
Transversals, themselves, define a matroid.

**Theorem 5.4.1**

*If \( \mathcal{V} \) is a family of finite subsets of a ground set \( V \), then the collection of partial transversals of \( \mathcal{V} \) is the set of independent sets of a matroid \( M = (V, \mathcal{V}) \) on \( V \).*
Transversals, themselves, define a matroid.

**Theorem 5.4.1**

*If $\mathcal{V}$ is a family of finite subsets of a ground set $V$, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on $V$.***

- This means that the transversals of $\mathcal{V}$ are the bases of matroid $M$. Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!
Transversals correspond exactly to matchings in bipartite graphs (as we’ve already strongly hinted at).
Transversals and Matchings

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- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$. 
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- A matching in this graph is a set of edges no two of which that have a common endpoint.
- In fact, we easily have

Lemma 5.4.2

A subset \(T \subseteq V\) is a partial transversal of \(\mathcal{V}\) iff there is a matching in \((V, I, E)\) in which every edge has one endpoint in \(T\).

We say that \(T\) is matched into \(I\).
Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$. 

\[ r(A) = \ell \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \] 

\[ = \ell \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \] 

\[ = \ell \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left\{ \begin{array}{ll} |A \cap V(J_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{array} \right\} + |I_i \setminus J_i| \] 

\[ = \ell \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap A| + |I_i \setminus J_i|) \]
Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$.

- We start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)$$  \hspace{1cm} (5.15)
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Morphing Partition Matroid Rank

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\[ r(A) = \sum_{i=1}^{\ell} \min(\{|A \cap V_i|, k_i\}) \quad (5.15) \]

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\[ = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left( \begin{array}{c} |A \cap V(I_i)| \quad \text{if } J_i \neq \emptyset \\ 0 \quad \text{if } J_i = \emptyset \end{array} \right) + |I_i \setminus J_i| \quad (5.17) \]
Morphing Partition Matroid Rank

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\]
Continuing,

\[ r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|) \quad (5.19) \]
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\[
\begin{align*}
    r(A) &= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|) \\
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\end{align*}
\]  

(5.19)  

(5.20)  

(5.21)  

(5.22)

In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.
In fact, we have

**Theorem 5.4.3**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

**Proof.**
Partial Transversals Are Matroids

In fact, we have

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- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus \((I1')\) holds.
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- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So \((I2')\) holds.
Partial Transversals Are Matroids

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**Theorem 5.4.3**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

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- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus \((I1')\) holds.

- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So \((I2')\) holds.

- Suppose that \(T_1\) and \(T_2\) are partial transversals of \(\mathcal{V}\) such that \(|T_1| < |T_2|\). Exercise: show that \((I3')\) holds.
Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  \hspace{1cm} (5.23)
Transversal Matroid Rank

Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  \hspace{1cm} (5.23)

Therefore, this function is submodular.
Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  

(5.23)

- Therefore, this function is submodular.

- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Matroid loops

- A circuit in a matroids is well defined, a subset \( A \subseteq E \) is **circuit** if it is an inclusionwise minimally dependent set (i.e., if \( r(A) < |A| \) and for any \( a \in A \), \( r(A \setminus \{a\}) = |A| - 1 \)).

- There is no reason in a matroid such an \( A \) could not consist of a single element.
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Such an $\{a\}$ is called a loop.
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- In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1. The 0 can appear $> 1$ time with different indices, as can a self loop in a graph appear on different nodes.
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- In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1. The 0 can appear $\geq 1$ time with different indices, as can a self loop in a graph appear on different nodes.

- Note, we also say that two elements $s, t$ are said to be **parallel** if $\{s, t\}$ is a circuit.
Definition 5.5.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).
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- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
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- We can more generally define matroids on a field.
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- We can more generally define matroids on a field.

Definition 5.5.2 (linear matroids on a field)

Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $X_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of $E$ such that the columns of $X$ are linearly independent over $\mathbb{F}$.
Definition 5.5.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
- We can more generally define matroids on a field.

Definition 5.5.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called **representable over** $F$. 

Prof. Jeff Bilmes
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In particular:

**Theorem 5.5.4**

*Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.*
The converse is not true, however.

**Example 5.5.5**

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$.
Converse: Representability of Transversal Matroids

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Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$.

- It can be shown that this is a matroid and is representable.
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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.
Definition 5.6.9 (closed/flat/subspace)
A subset \( A \subseteq E \) is closed or a flat or a subspace of matroid \( M \) if for all \( x \in E \setminus A \), \( r(A \cup \{x\}) = r(A) + 1 \).

Definition 5.6.10 (closure)
Given \( A \subseteq E \), the closure (or span) of \( A \), is defined by
\[
\text{span}(A) = \{ b \in E : r(A \cup \{b\}) = r(A) \}.
\]

Therefore, a closed set \( A \) has \( \text{span}(A) = A \).

Definition 5.6.11 (circuit)
A subset \( A \subseteq E \) is circuit or a cycle if it is an inclusionwise minimally dependent set (i.e., if \( r(A) < |A| \) and for any \( a \in A \), \( r(A \setminus \{a\}) = |A| - 1 \)).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.
Spanning Sets

- Consider terminology: spanning tree
Spanning Sets

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**Definition 5.6.1 (spanning set of a set)**

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a **spanning set of** $Y$. 
Spanning Sets

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Definition 5.6.2 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.
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Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
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**Definition 5.6.2 (spanning set of a matroid)**

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- $V$ is always trivially spanning.
Dual of a Matroid

- Given a matroid \( M = (V, \mathcal{I}) \), a dual matroid \( M^* \) can be defined in a way such that \((M^*)^* = M\).
Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.
- We define a set

$$
\mathcal{I}^* = \{ I \subseteq V : V \setminus I \text{ is a spanning set of } M \} \quad (5.24)
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Dual of a Matroid

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- Recall, in cycle matroid of a graph, a spanning set of \( G \) is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
Dual of a Matroid

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- Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
- Since the smallest spanning sets are bases, the bases of $M$ (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of $M^*$ (where $I$ is as large as possible).
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in $M$, so must $V \setminus I$. Therefore, (I2') holds.
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is a base in $M^*$, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in $M$. That is, removing $v$ from $V \setminus I$ doesn’t make $(V \setminus I) \setminus v$ not spanning.
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Since $V \setminus J$ is spanning, $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$. 

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- Since $V \setminus J$ is spanning, $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$.

- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can find a base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$. 

\[ \cdots \]
Theorem 5.6.3

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Proof.

- Consider $I, J \in I^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is a base in $M^*$, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in $M$. That is, removing $v$ from $V \setminus I$ doesn’t make $(V \setminus I) \setminus v$ not spanning.

- Since $V \setminus J$ is spanning, $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$.

- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can find a base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

- Since $B$ and $J$ are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, $B'$ and $I$ are disjoint.
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$$|B| = |B \cap I| + |B \setminus I|$$  \hspace{1cm} (5.25)

$$\leq |I \setminus J| + |B \setminus I|$$  \hspace{1cm} (5.26)

$$< |J \setminus I| + |B \setminus I| \leq |B'|$$  \hspace{1cm} (5.27)

which is a contradiction. The last inequality on the right follows since $J \setminus I \subseteq B'$ (by assumption) and $B \setminus I \subseteq B'$ implies that $J \setminus I \cup B \setminus I \subseteq B'$, but since $J$ and $B$ are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq B'$. 

Prof. Jeff Bilmes

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Dual of a Matroid

Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

\[
|B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| \leq |J \setminus I| + |B \setminus I| \leq |B'|.
\]

which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
Dual of a Matroid

**Theorem 5.6.3**

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

**Proof.**

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

  $$|B| = |B \cap I| + |B \setminus I|$$  \hspace{2cm} (5.25)

  $$\leq |I \setminus J| + |B \setminus I|$$  \hspace{2cm} (5.26)

  $$< |J \setminus I| + |B \setminus I| \leq |B'|$$  \hspace{2cm} (5.27)

  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.

- So $B'$ is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in $M$, and therefore $I \cup \{v\} \in \mathcal{I}^*$. 
Dual Matroid Rank

**Theorem 5.6.4**

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (5.28)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *i.e.*, $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
Dual Matroid Rank

Theorem 5.6.4

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified as follows, for \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)
\] (5.28)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since

\[
|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).
\]

The right inequality follows since \( r_M \) is submodular.
**Theorem 5.6.4**

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (5.28)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
Theorem 5.6.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (5.28)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, $r_{M^*}$ is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
Dual Matroid Rank

Theorem 5.6.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (5.28)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (5.29)

...
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (5.28)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (5.29)

or

$$r_M(V \setminus X) = r_M(V)$$  \hspace{1cm} (5.30)
Theorem 5.6.4

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified as follows, for \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)
\]  

(5.28)

Proof.

A set \( X \) is independent in \( (V, r_{M^*}) \) if and only if

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|
\]  

(5.29)

or

\[
r_M(V \setminus X) = r_M(V)
\]  

(5.30)

But a subset \( X \) is independent in \( M^* \) only if \( V \setminus X \) is spanning in \( M \) (by the definition of the dual matroid).
Sources for Today’s Lecture
