Announcements, Assignments, and Reminders

- Reminder: class web page is at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)

- Please do use our discussion board (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.
Outstanding Reading

- Read chapter 1 from Fujishige book.
- Read over lecture slides, all posted on our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/).
- See the summary slide at the end for some additional ideas for reading. A good summary of matroid properties is http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
Submodular Concave

- Why do we call the \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \) definition of submodularity, submodular concave?

- A continuous twice differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) is concave iff \( \nabla^2 f \preceq 0 \) (the Hessian matrix is nonpositive definite).

- Define a “discrete derivative” or difference operator defined on discrete functions \( f : 2^V \to \mathbb{R} \) as follows:

  \[
  (\nabla_B f)(A) \triangleq f(A \cup B) - f(A \setminus B) = f(B | (A \setminus B)) \tag{5.1}
  \]

  read as: the derivative of \( f \) at \( A \) in the direction \( B \).

- Hence, if \( A \cap B = \emptyset \), then \( (\nabla_B f)(A) = f(B | A) \).

- Consider a form of second derivative or 2nd difference:

  \[
  (\nabla_C \nabla_B f)(A) = \nabla_C [f(A \cup B) - f(A \setminus B)] \\
  = f(A \cup B \cup C) - f((A \cup C) \setminus B) - f((A \setminus C) \cup B) + f((A \setminus C) \setminus B) \tag{5.2}
  \]
If the second difference operator everywhere nonpositive:

\[ f(A \cup B \cup C) - f((A \cup C) \setminus B) \]
\[ - f((A \setminus C) \cup B) + f(A \setminus C \setminus B) \leq 0 \]  

(5.1)

then we have the equation:

\[ f((A \cup C) \setminus B) + f((A \setminus C) \cup B) \geq f(A \cup B \cup C) + f(A \setminus C \setminus B) \]  

(5.2)

Define \( A' = (A \cup C) \setminus B \) and \( B' = (A \setminus C) \cup B \). Then the above implies:

\[ f(A') + f(B') \geq f(A' \cup B') + f(A' \cap B') \]  

(5.3)

and note that \( A' \) and \( B' \) so defined can be arbitrary.

One sense in which submodular functions are like concave functions.
Submodular Concave

(a) $A' = (A \cup C) \setminus B$

(b) $B' = (A \setminus C) \cup B$

Figure: A figure showing $A' \cup B' = A \cup B \cup C$ and $A' \cap B' = A \setminus C \setminus B$. 
Figure: A figure showing $A' \cup B' = A \cup B \cup C$ and $A' \cap B' = A \setminus C \setminus B$. 

(c) $A' = (A \cup C) \setminus B$

(d) $B' = (A \setminus C) \cup B$
Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then $J$ is said to be an independent set.

**Definition 5.2.29 (Matroid)**

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$. 
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this yesterday, but it is possible)
Matroids, other definitions

Definition 5.2.34 (closed/flat/subspace)

A subset $A \subseteq E$ is closed or a flat or a subspace of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition 5.2.35 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by
$$\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}.$$ 

Therefore, a closed set $A$ has $\text{span}(A) = A$.

Definition 5.2.36 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.
Partition Matroid

- Let $V$ be our ground set.

- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

  \[ I = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \]  

  (5.16)

  where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.

- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

- We’ll show that property (I3’) in Def. ?? holds. If $X, Y \in I$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
What is the partition matroid’s rank function?

A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  

(5.16)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (even modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
3. Sums of submodular functions are submodular.

\(r(A)\) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
System of Representatives

- We can view this as a bipartite graph. The groups of $V$ are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$, and $V = (V_1, V_2, \ldots, V_6)$
  $= (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\})$. 

Diagram:

![Diagram of a bipartite graph with color tags and right neighbors.](image_url)
We can view this as a bipartite graph. The groups of $V$ are marked by color tags on the left, and also via right neighbors in the graph.

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A system of representatives would make sure that there is a representative for each color group. For example,
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The representatives are shown as colors on the left.
System of Representatives

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- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives are shown as colors on the left.
- Here, the set of representatives is not distinct. In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
System of Distinct Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)). Hence, \(|I| = |\mathcal{V}|\).
- A family \((v_i : i \in I)\) with \(v_i \in V\) for index set \(I\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. Let's restate (and rename) this as a:

**Definition 5.2.36 (transversal)**

Given a set system \((V, \mathcal{V})\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[ x \in V_{\pi(x)} \text{ for all } x \in T \]  \hspace{1cm} (5.17)

- Note that due to it being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct).
A set $X \subseteq V$ is a **partial transversal** if $X$ is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.
A set \( X \subseteq V \) is a **partial transversal** if \( X \) is a transversal of some subfamily \( \mathcal{V}' = \{ V_i : i \in I' \} \) where \( I' \subseteq I \).

Therefore, for any transversal \( T \), any subset \( T' \subseteq T \) is a partial transversal (down closed).
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[
V(J) = \bigcup_{j \in J} V_j
\]

so \(|V(J)|\) is the set cover function.
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- We have

**Theorem 5.3.1 (Hall’s theorem)**

*Given a set system \((V, \mathcal{V})\), the family of subsets \(\mathcal{V} = (V_i : i \in I)\) has a transversal \((v_i : i \in I)\) iff for all \(J \subseteq I\)*

\[
|V(J)| \geq |J|
\]  

(5.2)
When do transversals exist?

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- Hall’s theorem as a bipartite graph.
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When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all $i$. Then, for any $J \subseteq I$, let
  \[ V(J) = \bigcup_{j \in J} V_j \]  
  (5.1)

  so $|V(J)|$ is the set cover function.
- Moreover, we have

**Theorem 5.3.2 (Rado’s theorem)**

If $M = (V, r)$ is a matroid on $V$ with rank function $r$, then the family of subsets $(V_i : i \in I)$ of $V$ has a transversal $(v_i : i \in I)$ which is independent in $M$ iff for all $J \subseteq I$

\[ r(V(J)) \geq |J| \]  
(5.3)
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[ V(J) = \bigcup_{j \in J} V_j \tag{5.1} \]

so \(|V(J)|\) is the set cover function.
- Moreover, we have

**Theorem 5.3.2 (Rado’s theorem)**

If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) which is independent in \(M\) iff for all \(J \subseteq I\)

\[ r(V(J)) \geq |J| \tag{5.3} \]

- Note, a transversal \(T\) independent in \(M\) means that \(r(T) = |T|\).
More general conditions for existence of transversals

**Theorem 5.3.3**

If \( \mathcal{V} = (V_i : I \in I) \) is a finite family of non-empty subsets of \( V \), and \( f : 2^V \to \mathbb{Z}_+ \) is a non-negative, integral, monotone non-decreasing, and submodular function, then \( \mathcal{V} \) has a system of representatives \((v_i : i \in I)\) such that

\[
f(\bigcup_{i \in J} \{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{5.4}
\]

if and only if

\[
f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{5.5}
\]
More general conditions for existence of transversals

Theorem 5.3.3

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \rightarrow \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\bigcup_{i \in J} \{v_i\}) \geq |J| \text{ for all } J \subseteq I$$

(5.4)

if and only if

$$f(\mathcal{V}(J)) \geq |J| \text{ for all } J \subseteq I$$

(5.5)

Given Theorem 5.3.3, we immediately get Theorem 5.3.1 by taking $f(S) = |S|$ for $S \subseteq V$. In which case, Eq. 5.4 requires the system of representatives to be distinct.
More general conditions for existence of transversals

**Theorem 5.3.3**

If \( \mathcal{V} = (V_i : I \in I) \) is a finite family of non-empty subsets of \( V \), and \( f : 2^V \to \mathbb{Z}_+ \) is a non-negative, integral, monotone non-decreasing, and submodular function, then \( \mathcal{V} \) has a system of representatives \((v_i : i \in I)\) such that

\[
f(\bigcup_{i \in J} \{v_i\}) \geq |J| \quad \text{for all} \ J \subseteq I \tag{5.4}
\]

if and only if

\[
g(\emptyset) \leq f(\mathcal{V}(J)) - |J| \quad \text{for all} \ J \subseteq I \tag{5.5}
\]

Given Theorem 5.3.3, we immediately get Theorem 5.3.1 by taking

\[
f(S) = |S| \quad \text{for} \ S \subseteq V.
\]

We get Theorem 5.3.2 by taking \( f(S) = r(S) \) for \( S \subseteq V \), the rank function of the matroid. \textit{where, Eq. 5.4 insists the system of representatives is independent in} \( M \).
More general conditions for existence of transversals

first part proof of Theorem 5.3.3.

Suppose Eq. 5.4 is true. Then since $f$ is monotone, and since $V(J) \supseteq \bigcup_{i \in J} \{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 5.5 immediately follows.

...
Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| > 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

When Eq. 5.5 and the above holds, this means that for any subsets \( J_1, J_2 \subseteq I \setminus \{1\} \), we have that

\[
\begin{align*}
f(V_1 \cup V(J_1)) &\geq |J_1| + 1 \quad (5.6) \\
f(V_1 \cup V(J_2)) &\geq |J_2| + 1 \quad (5.7)
\end{align*}
\]
Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| > 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

Suppose, to the contrary, the consequent is false. Then we may take \( \bar{v}_1, \bar{v}_2 \in V_1 \) as two distinct elements in \( V_1 \),...
Lemma 5.3.4

Suppose Eq. 5.5 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V \), and there exists an \( i \) such that \( |V_i| > 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take \( \bar{v}_1, \bar{v}_2 \in V_1 \) as two distinct elements in \( V_1 \),
- and there must exist subsets \( J_1, J_2 \) of \( I \setminus \{1\} \) such that

\[
\begin{align*}
\text{for } J_1: & \quad f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) < |J_1| + 1, \quad (5.8) \\
\text{for } J_2: & \quad f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) < |J_2| + 1, \quad (5.9)
\end{align*}
\]

(note that either one or both of \( J_1, J_2 \) could be empty).
Lemma 5.3.4

Suppose Eq. 5.5 \( (f(V(J)) \geq |J|, \forall J \subseteq I) \) is true for \( V \), and there exists an \( i \) such that \( |V_i| > 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n) \) also satisfies Eq 5.5.

Proof.

- Taking \( X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1) \) and \( Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2) \), we have \( f(X) \leq |J_1|, f(Y) \leq |J_2| \), and that:

\[
X \cup Y = V_1 \cup V(J_1 \cup J_2), \quad (5.10)
\]
\[
X \cap Y \supseteq V(J_1 \cap J_2), \quad (5.11)
\]

and

\[
|J_1| + |J_2| \geq f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \quad (5.12)
\]
Lemma 5.3.4

Suppose Eq. 5.5 \((f(V(J)) \geq |J|, \forall J \subseteq I)\) is true for \(V\), and there exists an \(i\) such that \(|V_i| > 2\) (w.l.o.g., say \(i = 1\)). Then there exists \(\bar{v} \in V_1\) such that the family of subsets \((V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_n)\) also satisfies Eq 5.5.

Proof.

- since \(f\) submodular monotone non-decreasing, & Eqs 5.6-5.9,

\[|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \quad (5.13)\]
More general conditions for existence of transversals

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Proof.

- since \( f \) submodular monotone non-decreasing, & Eqs 5.6-5.9,

\[
|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{5.13}
\]

- Since \( V \) satisfies Eq. 5.5, \( 1 \notin J_1 \cup J_2 \), & Eqs 5.6-5.7, this gives

\[
|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{5.14}
\]

which is a contradiction since cardinality is modular.
converse proof of Theorem 5.3.3.

Conversely, suppose Eq. 5.5 is true.
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converse proof of Theorem 5.3.3.

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- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
More general conditions for existence of transversals

Converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ until we arrive at a family of singleton sets.
More general conditions for existence of transversals

converse proof of Theorem 5.3.3.

- Conversely, suppose Eq. 5.5 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 5.3.4, the family of subsets $(V_1 \setminus \{f\}, V_2, \ldots, V_n)$ also satisfies Eq 5.5.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ until we arrive at a family of singleton sets.
- This family will be the required system of representatives.
converse proof of Theorem 5.3.3.

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- This family will be the required system of representatives.

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.
Transversals, themselves, define a matroid.

**Theorem 5.4.1**

*If \( \mathcal{V} \) is a family of finite subsets of a ground set \( V \), then the collection of partial transversals of \( \mathcal{V} \) is the set of independent sets of a matroid \( M = (V, \mathcal{V}) \) on \( V \).*
Transversals, themselves, define a matroid.

**Theorem 5.4.1**

If $\mathcal{V}$ is a family of finite subsets of a ground set $V$, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on $V$.

- This means that the transversals of $\mathcal{V}$ are the bases of matroid $M$. Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!
Transversals and Matchings

- Transversals correspond exactly to matchings in bipartite graphs (as we’ve already strongly hinted at).
Transversals and Matchings

- Transversals correspond exactly to matchings in bipartite graphs (as we’ve already strongly hinted at).

- Given a set system \((V, \mathcal{V})\), with \(\mathcal{V} = (V_i : i \in I)\), we can define a bipartite graph \(G = (V, I, E)\) associated with \(\mathcal{V}\) that has edge set \(\{(v, i) : v \in V, i \in I, v \in V_i\}\).
Transversals and Matchings

- Transversals correspond exactly to matchings in bipartite graphs (as we’ve already strongly hinted at).
- Given a set system \((V, \mathcal{V})\), with \(\mathcal{V} = (V_i : i \in I)\), we can define a bipartite graph \(G = (V, I, E)\) associated with \(\mathcal{V}\) that has edge set \(\{(v, i) : v \in V, i \in I, v \in V_i\}\).
- A matching in this graph is a set of edges no two of which have a common endpoint.
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- A matching in this graph is a set of edges no two of which have a common endpoint.
- In fact, we easily have

**Lemma 5.4.2**

A subset \(T \subseteq V\) is a partial transversal of \(\mathcal{V}\) iff there is a matching in \((V, I, E)\) in which every edge has one endpoint in \(T\).

We say that \(T\) is matched into \(I\).
Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$. 

\[ r(A) = \ell \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  
\[ = \ell \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \]  
\[ = \ell \sum_{i=1}^{\ell} \min J_{i} \subseteq I_i \left( |V(J_i) \cap A| + |I_i \setminus J_i| \right) \]
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We start with partition matroid rank function in the subsequent equations.

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r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{5.15}
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= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \tag{5.16}
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= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left( \left\{ \begin{array}{ll} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{array} \right\} + |I_i \setminus J_i| \right) 
\] (5.15, 5.16, 5.17)
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We start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)$$  \hspace{1cm} (5.15)

$$= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|)$$  \hspace{1cm} (5.16)

$$= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left( \begin{array}{ccc} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{array} \right) + |I_i \setminus J_i|$$  \hspace{1cm} (5.17)

$$= \sum_{i=1}^{\ell} \min (|V(J_i) \cap A| + |I_i \setminus J_i|)$$  \hspace{1cm} (5.18)
Continuing,\[ r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|) \] (5.19)
Continuing,

\[ r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left( |V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \]  

(5.19)

\[ = \min_{J \subseteq I} \left( \sum_{i=1}^{\frac{|I|}{|I_i|}} |V(J) \cap V(I_i) \cap A| - |I_i \cap J| + |I_i| \right) \]  

(5.20)
Continuing,

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In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.
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In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.
Partial Transversals Are Matroids

In fact, we have

**Theorem 5.4.3**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

**Proof.**
Partial Transversals Are Matroids

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Proof.

- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus (I1') holds.
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**Proof.**

- We note that \(\emptyset \in I\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus \((I1')\) holds.
- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So \((I2')\) holds.
In fact, we have

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**Proof.**

- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus (I1’) holds.
- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So (I2’) holds.
- Suppose that \(T_1\) and \(T_2\) are partial transversals of \(\mathcal{V}\) such that \(|T_1| < |T_2|\). Exercise: show that (I3’) holds.
Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  

(5.23)
Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  \hspace{1cm} (5.23)

- Therefore, this function is submodular.
Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  

(5.23)

- Therefore, this function is submodular.

- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

There is no reason in a matroid such an $A$ could not consist of a single element.
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Such an $\{a\}$ is called a loop.
A circuit in a matroids is well defined, a subset \( A \subseteq E \) is **circuit** if it is an inclusionwise minimally dependent set (i.e., if \( r(A) < |A| \) and for any \( a \in A \), \( r(A \setminus \{a\}) = |A| - 1 \)).

There is no reason in a matroid such an \( A \) could not consist of a single element.

Such an \( \{a\} \) is called a **loop**.

In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1.
A circuit in a matroids is well defined, a subset $A \subseteq E$ is **circuit** if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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Such an $\{a\}$ is called a **loop**.

In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1.

Note, we also say that two elements $s, t$ are said to be **parallel** if $\{s, t\}$ is a circuit.
Definition 5.5.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).
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Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as $GF(p)$ where $p$ is prime (such as $GF(2)$).
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- We can more generally define matroids on a field.

Definition 5.5.2 (linear matroids on a field)

Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $X_{ij} \in \mathbb{F}$ for some field, and let $I$ be the set of subsets of $E$ such that the columns of $X$ are linearly independent over $\mathbb{F}$.
Definition 5.5.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).
- We can more generally define matroids on a field.

Definition 5.5.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called **representable over** $\mathbb{F}$.
Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
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In particular:

**Theorem 5.5.4**

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.
The converse is not true, however.

Example 5.5.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$. It can be shown that this is a matroid and is representable. However, this matroid is not isomorphic to any transversal matroid.
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**Example 5.5.5**

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.
Matroids, other definitions

**Definition 5.6.34 (closed/flat/subspace)**

A subset $A \subseteq E$ is closed or a flat or a subspace of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

**Definition 5.6.35 (closure)**

Given $A \subseteq E$, the closure (or span) of $A$, is defined by

$$\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}.$$

Therefore, a closed set $A$ has $\text{span}(A) = A$.

**Definition 5.6.36 (circuit)**

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.
Spanning Sets

- Consider terminology: spanning tree
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**Definition 5.6.1 (spanning set of a set)**

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a **spanning set of $Y$**.
Spanning Sets

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Given a matroid \( M = (V, \mathcal{I}) \), and a set \( Y \subseteq V \), then any set \( X \subseteq Y \) such that \( r(X) = r(Y) \) is called a spanning set of \( Y \).

**Definition 5.6.2 (spanning set of a matroid)**

Given a matroid \( M = (V, \mathcal{I}) \), any set \( A \subseteq V \) such that \( r(A) = r(V) \) is called a spanning set of the matroid.
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**Definition 5.6.2 (spanning set of a matroid)**

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a **spanning set** of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
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**Definition 5.6.2 (spanning set of a matroid)**

Given a matroid $\mathcal{M} = (V, I)$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- $V$ is always trivially spanning.
Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$. 
Given a matroid $\mathcal{M} = (V, \mathcal{I})$, a dual matroid $\mathcal{M}^*$ can be defined in a way such that $(\mathcal{M}^*)^* = \mathcal{M}$.

We define a set

$$\mathcal{I}^* = \{ I \subseteq V : V \setminus I \text{ is a spanning set of } \mathcal{M} \} \quad (5.24)$$
Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.
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- Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
Dual of a Matroid

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Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).

Since the smallest spanning sets are bases, the bases of $M$ (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of $M^*$ (where $I$ is as large as possible).
Dual of a Matroid

Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
Theorem 5.6.3

Let \( M^* \) be defined as on previous slide. Then \( M^* \) is a matroid.

Proof.

- Clearly \( \emptyset \in I^* \), so (I1’) holds.

- Also, if \( I \subseteq J \in \mathcal{I}^* \), then clearly also \( I \in \mathcal{I}^* \) since if \( V \setminus J \) is spanning in \( M \), so must \( V \setminus I \). Therefore, (I2’) holds.
Theorem 5.6.3

Let \( M^* \) be defined as on previous slide. Then \( M^* \) is a matroid.

Proof.

- Consider \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \).
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$. 

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Theorem 5.6.3

Let \( M^* \) be defined as on previous slide. Then \( M^* \) is a matroid.

Proof.

1. Consider \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \).
2. \( V \setminus J \) contain some base (say \( B \subseteq V \setminus J \)) of \( M \). Also, \( V \setminus I \) contains a base of \( M \).
3. Since \( B \setminus I \subseteq V \setminus I \), and \( B \setminus I \) is independent in \( M \), we can find a base \( B' \) of \( M \) s.t. \( B \setminus I \subseteq B' \subseteq V \setminus I \).
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Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.
- $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$.
- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can find a base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since $B$ and $J$ are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, $B'$ and $I$ are disjoint.
Theorem 5.6.3

Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

Proof.

Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$$\begin{align*}
|B| &= |B \cap I| + |B \setminus I| \\
&\leq |I \setminus J| + |B \setminus I| \\
&< |J \setminus I| + |B \setminus I| \leq |B'|
\end{align*}$$

which is a contradiction. The last inequality on the right follows since $J \setminus I \subseteq B'$, and $B \setminus I \subseteq B'$ implies that $J \setminus I \cup B \setminus I \subseteq B'$, but since $J$ and $B$ are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq B'$. 


Let $M^*$ be defined as on previous slide. Then $M^*$ is a matroid.

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$$|B| = |B \cap I| + |B \setminus I|$$  \hspace{1cm} (5.25)

$$\leq |I \setminus J| + |B \setminus I|$$  \hspace{1cm} (5.26)

$$< |J \setminus I| + |B \setminus I| \leq |B'|$$  \hspace{1cm} (5.27)

which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$. 

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\[
|B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| < |J \setminus I| + |B \setminus I| \leq |B'|
\]

which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.

- So $B'$ is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in $M$, and therefore $I \cup \{v\} \in \mathcal{I}^*$.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (5.28)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
Theorem 5.6.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{5.28}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since

  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$

  The right inequality follows since $r_M$ is submodular.
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- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
Theorem 5.6.4

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified as follows, for \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{5.28}
\]

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  \[
  |X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).
  \]
- Monotone non-decreasing follows since, as \( X \) increases by one, \( |X| \) always increases by 1, while \( r_M(V \setminus X) \) decreases by one or zero.
- Therefore, \( r_{M^*} \) is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
Theorem 5.6.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (5.28)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hspace{1cm} (5.29)
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or

$$r_M(V \setminus X) = r_M(V)$$  \hspace{1cm} (5.30)
Dual Matroid Rank

Theorem 5.6.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

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or

$$r_M(V \setminus X) = r_M(V)$$  \hspace{1cm} (5.30)

But a subset $X$ is independent in $M^*$ only if $V \setminus X$ is spanning in $M$ (by the definition of the dual matroid).
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.
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The dual of the cycle matroid is called the cocycle matroid. It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.
Matroid and the greedy algorithm

- Let \( \mathcal{I} \) be a set of subsets of \( E \) that is down-closed. Consider a non-negative modular weight function \( w : E \to \mathbb{R}_+ \), and we want to find the \( A \in \mathcal{I} \) that maximizes \( w(A) \).

- Greedy algorithm: Set \( A = \emptyset \), and repeatedly choose \( y \in E \setminus A \) such that \( A \cup \{y\} \in \mathcal{I} \) with \( w(y) \) as large as possible, stopping when no such \( y \) exists.

**Theorem 5.7.1**

Let \( \mathcal{I} \) be a non-empty collection of subsets of a set \( E \), down-closed (i.e., an independence system). Then the pair \((E, \mathcal{I})\) is a matroid if and only if for each weight function \( w \in \mathcal{R}_+^E \), the greedy algorithm leads to a set \( I \in \mathcal{I} \) of maximum weight \( w(I) \).
Recall: Matroids by bases

**Theorem 5.7.2**

**Matroid (by bases)** Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
proof of Theorem 5.7.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathcal{R}_+\) is given.
proof of Theorem 5.7.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
proof of Theorem 5.7.1.

- Assume \((E, I)\) is a matroid and \(w : E \rightarrow \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.
proof of Theorem 5.7.1.

- Assume $(E, I)$ is a matroid and $w : E \rightarrow \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \ldots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).
- $A$ is a base of $M$, and let $B = (b_1, \ldots, b_r)$ be any another base of $M$ with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all $i$. 

...
proof of Theorem 5.7.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$. 

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- Assume otherwise, and let \( k \) be the first (smallest) integer such that \( w(a_k) < w(b_k) \).
- Define independent sets \( A_{k-1} = \{a_1, \ldots, a_{k-1}\} \) and \( B_k = \{b_1, \ldots, b_k\} \).
- Since \( |A_{k-1}| < |B_k| \), \( A_{k-1} \cup \{b_i\} \in \mathcal{I} \) for some \( 1 \leq i \leq k \).
proof of Theorem 5.7.1.

Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$.

Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.

Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen $b_i$ rather than $a_k$, contradicting what greedy does.
converse proof of Theorem 5.7.1.

- Given an independence system \((E, \mathcal{I})\), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \((E, \mathcal{I})\) is a matroid.
converse proof of Theorem 5.7.1.

- Given an independence system \( (E, \mathcal{I}) \), suppose the greedy algorithm leads to an independent set of max weight for each such weight function. We’ll show \( (E, \mathcal{I}) \) is a matroid.

- Down monotonicity already holds (since we’ve started with an independence system).
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- Let \(I, J \in \mathcal{I}\) with \(|I| < |J|\). Suppose to the contrary, that \(I \cup \{z\} \notin \mathcal{I}\) for all \(z \in J \setminus I\).
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- Define the following modular weight function \(w\) on \(V\), and define \(k = |I|\).

\[
    w(v) = \begin{cases} 
        k + 2 & \text{if } v \in I, \\
        k + 1 & \text{if } v \in J \setminus I, \\
        0 & \text{if } v \in S \setminus (I \cup J) 
    \end{cases}
\]  

(5.31)
converse proof of Theorem 5.7.1.

- Now greedy will clearly, after \( k \) iterations recover \( I \), but cannot choose any element in \( J \setminus I \) by assumption. Thus, greedy chooses a set of weight \( k(k+2) \).
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- On the other hand, $J$ has weight

$$w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2) \tag{5.32}$$

so $J$ has strictly larger weight but is still independent, contradicting greedy’s optimality.
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so $J$ has strictly larger weight but is still independent, contradicting greedy’s optimality.

- Therefore, $(E, I)$ must be a matroid.
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

(5.33)

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. 

This is called the restriction of $M$ to $Y$, and is often written $M|_Y$.

If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\}$$

(5.34)

is considered a deletion of $X$ from $M$, and is often written $M \setminus X$.

The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$. 

Prof. Jeff Bilmes
EE596A/Fall 2012/Submodularity – Lecture 5 - October 10th, 2012
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Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** $Z$ is written $M/Z$. 

\[ r_{M/Z}(X) = r(X \cup Z) - r(Z) = \rho_X(Z) = r(X|Z) \quad (5.35) \] 

(see equations 57-60 from lecture 2). A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid contraction

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- We can see that $M/Z = (M^* \setminus Z)^*$. 
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Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$. 

Theorem 5.8.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$\min_{X \subseteq V} (r_1(X) + r_2(V \setminus X)) \quad (5.36)$$

In general, this is an instance of the convolution of two submodular functions, which more generally is written as:

$$(r_1 \ast r_2)(Y) = \min_{X \subseteq Y} (r_1(X) + r_2(Y \setminus X)) \quad (5.37)$$
Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$. 

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While $(V, I_1 \cap I_2)$ is typically not a matroid, we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$.

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$$ (r_1 \ast r_2)(Y) = \min_{X \subseteq Y} \left( r_1(X) + r_2(Y \setminus X) \right) \quad (5.37) $$
Definition 5.8.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\ldots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

$$M_1 \vee M_2 \vee \cdots \vee M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k),$$

where

$$\mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k = \{ I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k \} \quad (5.38)$$
**Matroid Union**

**Definition 5.8.2**

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

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$$I_1 \vee I_2 \vee \cdots \vee I_k = \{I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\} \quad (5.38)$$

**Theorem 5.8.3**

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left( |Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \right) \quad (5.39)$$

for any $Y \subseteq V_1 \cup \ldots \cup V_k$. 
Sources for Today’s Lecture
