EE596A – Submodularity Functions, Optimization, and Application to Machine Learning
Fall Quarter 2012

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http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/

Lecture 6 - October 12th, 2012
Reminder: class web page is at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/)

Please do use our discussion board (https://catalyst.uw.edu/gopost/board/bilmes/29948/) for all questions, comments, so that all will benefit from them being answered.
Read chapter 1 from Fujishige book.

Read chapter 2 from Fujishige book.

Read over lecture slides, all posted on our web page (http://j.ee.washington.edu/~bilmes/classes/ee596a_fall_2012/).

See the summary slide at the end for some additional ideas for reading. A good summary of matroid properties is http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then $J$ is said to be an independent set.

**Definition 6.2.4 (Matroid)**

A set system $(E, \mathcal{I})$ is a **Matroid** if

1. $\emptyset \in \mathcal{I}$
2. $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$
3. $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$. 
Matroid

Slight modification (non unit increment) that is equivalent.

Definition 6.2.4 (Matroid-II)

A set system \((E, I)\) is a Matroid if

(I1') \(\emptyset \in I\)

(I2') \(\forall I \in I, J \subset I \Rightarrow J \in I\) (or “down-closed”)

(I3') \(\forall I, J \in I, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in I\)

Note (I1)=(I1'), (I2)=(I2'), and we get (I3)\(\equiv\)(I3') using induction.
Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^*$ can be defined in a way such that $(M^*)^* = M$.
- We define a set

$$\mathcal{I}^* = \{I \subseteq V : V \setminus I \text{ is a spanning set of } M\} \quad (6.24)$$

- Recall, in cycle matroid of a graph, a spanning set of $G$ is any set of edges that are adjacent to all nodes (i.e., any superset of a spanning forest).
- Since the smallest spanning sets are bases, the bases of $M$ (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of $M^*$ (where $I$ is as large as possible).
Dual Matroid Rank

Theorem 6.2.24

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified as follows, for $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (6.27)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, $r_{M^*}$ is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
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**Proof.**

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \quad (6.28)$$
Dual Matroid Rank

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(6.28)

or

$$r_M(V \setminus X) = r_M(V)$$

(6.29)

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**Dual Matroid Rank**

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But a subset $X$ is independent in $M^*$ only if $V \setminus X$ is spanning in $M$ (by the definition of the dual matroid).
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
Example duality: cocycle matroid

- The dual of the cycle matroid is called the cocycle matroid.
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

A graph $G$
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Minimally spanning in $M$ (and thus a base in $M$)

Minimally spanning in $M^*$ (and thus a base in $M^*$)
Example duality: cocycle matroid

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Independent but not spanning in $M$

Dependent in $M^*$ (contains a cocycle, is a nonminimal cut)
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Spanning in $M$, but not a base

Independent in $M^*$
Example duality: cocycle matroid

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Independent but not spanning in $M$

A cycle in $M^*$ (a cocycle, or a minimal cut)
Matroid and the greedy algorithm

- Let $\mathcal{I}$ be a set of subsets of $E$ that is down-closed. Consider a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.

- Consider the greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that: 1) $A \cup \{y\} \in \mathcal{I}$, and 2) $w(y)$ is as large as possible. We stop when no such $y$ exists.

**Theorem 6.4.1**

Let $\mathcal{I}$ be a non-empty collection of subsets of a set $E$, down-closed (i.e., an independence system). Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$. 
Theorem 6.4.9 (Matroid (by bases))

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
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Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
proof of Theorem 6.4.1.

• Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathcal{R}_+\) is given.
proof of Theorem 6.4.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathcal{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
Matroid and the greedy algorithm

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- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.

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- Assume $(E, I)$ is a matroid and $w: E \rightarrow \mathbb{R}_+$ is given.
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- $A$ is a base of $M$, and let $B = (b_1, \ldots, b_r)$ be any another base of $M$ with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all $i$. 

...
Matroid and the greedy algorithm

proof of Theorem 6.4.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$. 

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- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.
- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$. 

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.
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- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.
- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.
- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen $b_i$ rather than $a_k$, contradicting what greedy does.
Matroid and the greedy algorithm

Converse proof of Theorem 6.4.1.

- Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for every such weight function. We’ll show \((E, I)\) is a matroid.
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- Down monotonicity already holds (since we’ve started with an independence system).
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- Down monotonicity already holds (since we’ve started with an independence system).

- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$. 

Define the following modular weight function $w$ on $V$, and define $k = |I|$.

$$w(v) = \begin{cases} 
  k + 2 & \text{if } v \in I, \\
  k + 1 & \text{if } v \in J \setminus I, \\
  0 & \text{if } v \in S \setminus (I \cup J) 
\end{cases}$$

(6.1)
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  \end{cases}
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converse proof of Theorem 6.4.1.

Now greedy will clearly, after $k$ iterations recover $I$, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2)$.

On the other hand, $J$ has weight $w(J) \geq |J|(k+1) \geq (k+1)(k+1) > k(k+2)$ (6.2), so $J$ has strictly larger weight but is still independent, contradicting greedy's optimality. Therefore, $(E, I)$ must be a matroid.
Matroid and the greedy algorithm

**Converse proof of Theorem 6.4.1.**

- Now greedy will clearly, after $k$ iterations recover $I$, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2)$.
- On the other hand, $J$ has weight

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Therefore, $(E, \mathcal{I})$ must be a matroid.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. 
Matroid and greedy

- As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. 
- This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.
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This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.

If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. This will not only return an independent set, but it will return a base if we keep going even if the weights are 0. If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set. We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. This will not only return an independent set, but it will return a base if we keep going even if the weights are 0. If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set. We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base. If we stop at a negative value, we’ll once again get a maximum weight independent set.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. This will not only return an independent set, but it will return a base if we keep going even if the weights are 0. If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.

We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.

If we stop at a negative value, we’ll once again get a maximum weight independent set.

We can instead do **as small as possible** thus giving us a minimum weight independent set/base.
Let \( M = (V, \mathcal{I}) \) be a matroid and let \( Y \subseteq V \), then

\[
\mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \}
\]

is such that \( M_Y = (Y, \mathcal{I}_Y) \) is a matroid with rank \( r(M_Y) = r(Y) \).
Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

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is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

- This is called the restriction of $M$ to $Y$, and is often written $M|_Y$. 
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

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is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. This is called the restriction of $M$ to $Y$, and is often written $M|Y$.

If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\} \quad (6.4)$$

is considered a deletion of $X$ from $M$, and is often written $M \setminus Z$. 
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

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is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the **restriction** of $M$ to $Y$, and is often written $M|Y$.

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Hence, $M|Y = M \setminus (V \setminus Y)$. 

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  is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.
- This is called the **restriction** of $M$ to $Y$, and is often written $M|Y$.
- If $Y = V \setminus X$, then we have
  \[
  \mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\}
  \] (6.4)
  is considered a **deletion** of $X$ from $M$, and is often written $M \setminus Z$.
- Hence, $M|Y = M \setminus (V \setminus Y)$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$. 
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$. 

Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.

In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).

The rank function takes the form $r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z)$ (6.5).

So given $I \subseteq V \setminus Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X| = r(I \cup X)$, so $I \cup X$ independent in $M$. 

A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid contraction

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- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.
- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).
- The rank function takes the form

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z) \quad (6.5)$$
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$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z) \quad (6.5)$$

- So given $I \subseteq V \setminus Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X| = r(I \cup X)$, so $I \cup X$ independent in $M$. 
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• The rank function takes the form

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z)$$  \hspace{1cm} (6.5)

• So given $I \subseteq V \setminus Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X| = r(I \cup X)$, so $I \cup X$ independent in $M$.

• A **minor** of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid Intersection

Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$. 

Theorem 6.5.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$
(r_1 \circ r_2)(V) \equiv \min_{X \subseteq V} (r_1(X) + r_2(V \setminus X))
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This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

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Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

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Recall Hall's theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$. 

Convolution and Hall's Theorem
Recall Hall’s theorem, that a transversal exists iff for all \( X \subseteq V \), we have \( |\Gamma(X)| \geq |X| \).

\[ \iff |\Gamma(X)| - |X| \geq 0, \forall X \]
Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.

$\iff |\Gamma(X)| - |X| \geq 0, \forall X$

$\iff \min_X |\Gamma(X)| - |X| \geq 0$
Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.

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$\iff \min_X |\Gamma(X)| + |V| - |X| \geq |V|$
Recall Hall’s theorem, that a transversal exists iff for all \( X \subseteq V \), we have \( |\Gamma(X)| \geq |X| \).

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\[ \iff [\Gamma(\cdot) \ast |\cdot|](V) \geq |V| \]
Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.

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So Hall’s theorem can be expressed as convolution.
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$\iff [\Gamma(\cdot) \ast |\cdot|](V) \geq |V|$

So Hall’s theorem can be expressed as convolution.

Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).
**Matroid Union**

**Definition 6.5.2**

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

$$M_1 \lor M_2 \lor \cdots \lor M_k = (V_1 \uplus V_2 \uplus \cdots \uplus V_k, \mathcal{I}_1 \lor \mathcal{I}_2 \lor \cdots \lor \mathcal{I}_k),$$

where

$$I_1 \lor I_2 \lor \cdots \lor I_k = \{I_1 \uplus I_2 \uplus \cdots \uplus I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\} \quad (6.8)$$

Note $A \uplus B$ designates the disjoint union of $A$ and $B$. 
Matroid Union

Definition 6.5.2

Let $M_1 = (V_1, I_1)$, $M_2 = (V_2, I_2)$, $\ldots$, $M_k = (V_k, I_k)$ be matroids. We define the union of matroids as

$M_1 \lor M_2 \lor \cdots \lor M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, I_1 \lor I_2 \lor \cdots \lor I_k)$, where

$I_1 \lor I_2 \lor \cdots \lor I_k = \{I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in I_1, \ldots, I_k \in I_k \}$ (6.8)

Note $A \uplus B$ designates the disjoint union of $A$ and $B$.

Theorem 6.5.3

Let $M_1 = (V_1, I_1)$, $M_2 = (V_2, I_2)$, $\ldots$, $M_k = (V_k, I_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left( |Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \right)$$ (6.9)

for any $Y \subseteq V_1 \cup \ldots V_k$. 
Exercise: Describe $M \lor M^*$. 
Matroids of three or fewer elements are graphic

- All matroids up to and including three elements are graphic.
All matroids up to and including three elements are graphic.

(a) The only matroid with zero elements.
(b) The two one-element matroids.
(c) The four two-element matroids.
(d) The eight three-element matroids.
Matroids of three or fewer elements are graphic

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- Nice way to show low element size matroids. What about matroids that are low rank but with many elements?
Affine Matroids

Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$ is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i v_i = 0$ and $\sum_{i=1}^{k} a_i = 0$, and otherwise affinely independent.
Affine Matroids

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- Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1$ are linearly independent.
Affine Matroids

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- Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1$ are linearly independent.

**Proposition 6.5.4 (affine matroid)**

Let ground set $E = \{1, \ldots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets $X$ of $E$ such that $X$ indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.

**Proof.**

**Exercise:**
Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with \( n \times m = 2 \times 6 \) matrix on the field \( \mathbb{F} = \mathbb{R} \), and let the elements be 
\[
\{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\}.
\]
Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be 
\{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\}.

Hence, we can plot the points in $\mathbb{R}^2$ as follows:
Consider the affine matroid with \( n \times m = 2 \times 6 \) matrix on the field \( \mathbb{F} = \mathbb{R} \), and let the elements be
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Dependent sets consist of all subsets with \( \geq 4 \) elements, or 3 collinear elements.
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Hence, we can plot the points in \( \mathbb{R}^2 \) as follows:

Dependent sets consist of all subsets with \( \geq 4 \) elements, or 3 collinear elements.

In general, for a matroid \( M \) of rank \( m + 1 \) with \( m \leq 3 \), then a subset \( X \) in a geometric representation in \( \mathbb{R}^m \) is dependent if:

1. \( |X| \geq 2 \) and the points are identical;
2. \( |X| \geq 3 \) and the points are collinear;
3. \( |X| \geq 4 \) and the points are coplanar; or
4. \( |X| \geq 5 \) and the points are in space.
Theorem 6.5.5

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$. 

As another example, on the right, a rank 4 matroid:

$A(0,0,0)$
$B(0,0,1)$
$C(0,1,1)$
$D(0,1,0)$
$E(1,1,0)$
$F(1,0,0)$

All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:

$\{(0,0,0), (0,1,0), (1,1,0), (1,0,0)\}$,

$\{(0,0,0), (0,0,1), (0,1,1), (0,1,0)\}$, and

$\{(0,0,1), (0,1,1), (1,1,0), (1,0,0)\}$. 


Theorem 6.5.5

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As another example on the right, a rank 4 matroid

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:
  - $\{(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 0)\}$,
  - $\{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 0)\}$, and
  - $\{(0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0)\}$.
Loops represented by a separate box indicating how many loops there are. Parallel elements indicated by a multiplicity next to a point.
Euclidean Representation of Low-rank Matroids

- Very useful for graphically depicting low-rank matrices but which still have rich structure. Also useful for answering questions.
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- Example: Is there a matroid that is not representable (i.e., not linear for some field)?
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- Very useful for graphically depicting low-rank matrices but which still have rich structure. Also useful for answering questions.
- Example: Is there a matroid that is not representable (i.e., not linear for some field)? Yes, consider the matroid

```
1 7 8 9
2 3
6 5 4
```

Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that \{7, 8, 9\} is dependent, hence requiring an additional line in the above.
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![Diagram of the non-Pappus matroid](image)

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Euclidean Representation of Low-rank Matroids: A test

Is this a matroid?

1. 2. 3
4
7
5
6
1 2 3
4
7
5
6

Check rank's submodularity: Let $X = \{1, 2, 3, 6, 7\}$, $Y = \{1, 4, 5, 6, 7\}$. So $r(X) = 3$, and $r(Y) = 3$, and $r(X \cup Y) = 4$, so we must have, by submodularity, that $r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2$.

However, from the diagram, we have that since 1, 6, 7 are distinct non-collinear points, we have that $r(X \cap Y) = 3$.

If we extend the line from 6-7 to 1, then is it a matroid? Hence, not all 2D or 3D graphs of points and lines are matroids.
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- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.
Euclidean Representation of Low-rank Matroids: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):
Euclidean Representation of Low-rank Matroids: Other Examples

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- Hence, lines (in 2D) may be curved and planes (in 3D) can be twisted.
Euclidean Rep. of Low-rank Matroids: Conditions

- rank-1 (resp. rank-2, rank-3) flats correspond to points (resp. lines, planes).
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- rank-1 (resp. rank-2, rank-3) flats correspond to points (resp. lines, planes).
- A set of parallel points (could be size 1) does not touch another set of parallel points (could be size 1).
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- any two distinct points lie on a line (often not drawn when only two)
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- If diagram has at most one plane, then any two distinct lines meet in at most one point.
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- every plane contains at least three non-collinear points (not dependent unless $> 3$)
- any three distinct non-collinear points lie on a plane
- If diagram has at most one plane, then any two distinct lines meet in at most one point.
- If diagram has more than one plane, then: 1) any two distinct planes meeting in more than two points do so in a line; 2) any two distinct lines meeting in a point do so in at most one point and lie in on a common plane; 3) any line not lying on a plane intersects it in at most one point.
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- any two distinct points lie on a line (often not drawn when only two)
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- Matroid of rank at most four (see Oxley 2011 for more details).
Convex Polyhedra

- Convex polyhedra a large topic, we will only draw what we need.
Convex Polyhedra

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**Definition 6.6.1**

A subset $P \subseteq \mathbb{R}^E$ is a polyhedron if there exists an $m \times n$ matrix $A$ and vector $b \in \mathbb{R}^E$ (for some $m \geq 0$) such that

$$P = \{ x : Ax \leq b \}$$

(6.10)
Convex Polyhedra

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**Definition 6.6.1**

A subset $P \subseteq \mathbb{R}^E$ is a **polyhedron** if there exists an $m \times n$ matrix $A$ and vector $b \in \mathbb{R}^E$ (for some $m \geq 0$) such that

$$P = \{x : Ax \leq b\} \quad (6.10)$$

- Thus, $P$ is intersection of finitely many affine halfspaces, which are of the form $a_i x \leq b_i$ where $a_i$ is a row vector and $b_i$ a real scalar.
A polytope is defined as follows

**Definition 6.6.2**

A subset $P \subseteq \mathbb{R}^E$ is a **polytope** if it is the convex hull of finitely many vectors in $\mathbb{R}^E$. That is, if $\exists$, $x_1, x_2, \ldots, x_k \in \mathbb{R}^E$ such that for all $x \in P$, there exits $\{\lambda_i\}$ with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0 \ \forall i$ with $x = \sum_i \lambda_i x_i$. 
Convex Polytope

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- We define the convex hull operator as follows:

$$\text{conv}(x_1, x_2, \ldots, x_k) \overset{\text{def}}{=} \left\{ \sum_{i=1}^k \lambda_i x_i : \forall i, \lambda_i \geq 0, \text{ and } \sum_i \lambda_i = 1 \right\}$$

(6.11)
A polytope can be defined in a number of ways, two of which include

**Theorem 6.6.3**

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- $P$ is the convex hull of a finite set of points.
- If it is a **bounded** intersection of halfspaces, that is there exists matrix $A$ and vector $b$ such that

$$P = \{x : Ax \leq b\} \quad (6.12)$$
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- $P$ is the convex hull of a finite set of points.

- If it is a *bounded* intersection of halfspaces, that is there exists matrix $A$ and vector $b$ such that

$$P = \{x : Ax \leq b\} \quad (6.12)$$

This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.
Theorem 6.6.4 (weak duality)

Let $A$ be a matrix and $b$ and $c$ vectors, then

$$\max \{c^T x | Ax \leq b\} \leq \min \{y^T b : y \geq 0, y^T A = c^T\}$$

(6.13)
Theorem 6.6.4 (weak duality)

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$$\max \{ c^T x \mid Ax \leq b \} \leq \min \{ y^T b : y \geq 0, y^T A = c^T \}$$

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Theorem 6.6.5 (strong duality)

Let $A$ be a matrix and $b$ and $c$ vectors, then

$$\max \{ c^T x \mid Ax \leq b \} = \min \{ y^T b : y \geq 0, y^T A = c^T \}$$

(6.14)
Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max \{ c^T x | x \geq 0, Ax \leq b \} = \min \{ y^T b | y \geq 0, y^T A \geq c^T \} \quad (6.15)$$
There are many ways to construct the dual. For example,

\[
\begin{align*}
\max \{ c^T x \mid x \geq 0, Ax \leq b \} &= \min \{ y^T b \mid y \geq 0, y^T A \geq c^T \} \quad (6.15) \\
\max \{ c^T x \mid x \geq 0, Ax = b \} &= \min \{ y^T b \mid y^T A \geq c^T \} \quad (6.16)
\end{align*}
\]
Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max \{c^T x | x \geq 0, Ax \leq b\} = \min \{y^T b | y \geq 0, y^T A \geq c^T\} \quad (6.15)$$

$$\max \{c^T x | x \geq 0, Ax = b\} = \min \{y^T b | y^T A \geq c^T\} \quad (6.16)$$

$$\min \{c^T x | x \geq 0, Ax \geq b\} = \max \{y^T b | y \geq 0, y^T A \leq c^T\} \quad (6.17)$$
Linear Programming duality forms

There are many ways to construct the dual. For example,

\[
\text{max} \{c^\top x \mid x \geq 0, Ax \leq b\} = \text{min} \{y^\top b \mid y \geq 0, y^\top A \geq c^\top\} \quad (6.15)
\]

\[
\text{max} \{c^\top x \mid x \geq 0, Ax = b\} = \text{min} \{y^\top b \mid y^\top A \geq c^\top\} \quad (6.16)
\]

\[
\text{min} \{c^\top x \mid x \geq 0, Ax \geq b\} = \text{max} \{y^\top b \mid y \geq 0, y^\top A \leq c^\top\} \quad (6.17)
\]

\[
\text{min} \{c^\top x \mid Ax \geq b\} = \text{max} \{y^\top b \mid y \geq 0, y^\top A = c^\top\} \quad (6.18)
\]
Linear Programming duality forms

How to form the dual in general? We quote V. Vazirani (2001)
Linear Programming duality forms

How to form the dual in general? We quote V. Vazirani (2001)

*Intuitively, why is [one set of equations] the dual of [another quite different set of equations]? In our experience, this is not the right question to be asked. As stated in Section 12.1, there is a purely mechanical procedure for obtaining the dual of a linear program. Once the dual is obtained, one can devise intuitive, and possibly physical meaningful, ways of thinking about it. Using this mechanical procedure, one can obtain the dual of a complex linear program in a fairly straightforward manner. Indeed, the LP-duality-based approach derives its wide applicability from this fact.*
Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \quad (6.19)$$
Recall, any vector \( x \in \mathbb{R}^E \) can be seen as a modular function, as for any \( A \subseteq E \), we have

\[
x(A) = \sum_{a \in A} x_a
\]  
(6.19)

Given an \( A \subseteq E \), define the the incidence vector \( 1_A \in \{0, 1\}^E \) on the unit hypercube as follows:

\[
1_A \overset{\text{def}}{=} \left\{ x \in \{0, 1\}^E : x_i = 1 \text{ iff } i \in A \right\}
\]  
(6.20)

equivalently,

\[
1_A(j) \overset{\text{def}}{=} \begin{cases} 
1 & \text{if } j \in A \\
0 & \text{if } j \notin A
\end{cases}
\]  
(6.21)
Slight modification (non unit increment) that is equivalent.

**Definition 6.7.4 (Matroid-II)**

A set system \((E, I)\) is a **Matroid** if

1. \((I1')\) \(\emptyset \in I\)
2. \((I2')\) \(\forall I \in I, J \subset I \Rightarrow J \in I\) (or “down-closed”)
3. \((I3')\) \(\forall I, J \in I, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in I\)

Note \((I1) \equiv (I1')\), \((I2) \equiv (I2')\), and we get \((I3) \equiv (I3')\) using induction.
Independence Polyhedra

- For each \( I \in \mathcal{I} \) of a matroid \( M = (E, \mathcal{I}) \), we can form the incidence vector \( 1_I \).
Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \quad (6.22)$$
Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I$.

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$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\}$$

(6.22)

- Now take the rank function $r$ of $M$, and define the following polyhedron:

$$P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\}$$

(6.23)
Independence Polyhedra

- For each \( I \in \mathcal{I} \) of a matroid \( M = (E, \mathcal{I}) \), we can form the incidence vector \( \mathbf{1}_I \).
- Taking the convex hull, we get the independent set polytope, that is

\[
P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\}
\]  

(6.22)

- Now take the rank function \( r \) of \( M \), and define the following polyhedron:

\[
P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\}
\]  

(6.23)

- Now, take any \( x \in P_{\text{ind. set}} \), then we have that \( x \in P_r^+ \) (or \( P_{\text{ind. set}} \subseteq P_r^+ \)). We show this next.
If \( x \in P_{\text{ind. set}} \), then

\[
x = \sum_{i} \lambda_i 1_{I_i}
\]

for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).
\[ P_{\text{ind. set}} \subseteq P^+_r \]

- If \( x \in P_{\text{ind. set}} \), then
  \[ x = \sum_i \lambda_i 1_{I_i} \]  
  (6.24)

  for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).

- Clearly, for such \( x \), \( x \geq 0 \).
If $x \in P_{\text{ind. set}}$, then

$$x = \sum_{i} \lambda_i 1_{I_i}$$  \hfill (6.24)

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Clearly, for such $x$, $x \geq 0$.

Now, for any $A \subseteq E$,

$$x(A) = x^T 1_A = \sum_{i} \lambda_i 1_{I_i}^T 1_A$$  \hfill (6.25)
If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i 1_{I_i}$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Clearly, for such $x$, $x \geq 0$.

Now, for any $A \subseteq E$,

$$x(A) = x^T 1_A = \sum_i \lambda_i 1_{I_i}^T 1_A$$

$$\leq \sum_i \lambda_i \max_{j: I_j \subseteq A} 1_{I_j}(E)$$
If \( x \in P_{\text{ind. set}} \), then

\[
x = \sum \lambda_i 1_{I_i}
\]

(6.24)

for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).

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Now, for any \( A \subseteq E \),

\[
x(A) = x^T 1_A = \sum \lambda_i 1_{I_i}^T 1_A
\]

(6.25)

\[
\leq \sum \lambda_i \max_{j : I_j \subseteq A} 1_{I_j}(E)
\]

(6.26)

\[
= \max_{j : I_j \subseteq A} 1_{I_j}(E)
\]

(6.27)
\[ P_{\text{ind. set}} \subseteq P^+_r \]

- If \( x \in P_{\text{ind. set}} \), then
  \[
  x = \sum_i \lambda_i \mathbf{1}_{I_i} \tag{6.24}
  \]
  for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).
- Clearly, for such \( x \), \( x \geq 0 \).
- Now, for any \( A \subseteq E \),
  \[
  x(A) = x^\top \mathbf{1}_A = \sum_i \lambda_i \mathbf{1}_{I_i}^\top \mathbf{1}_A \tag{6.25}
  \]
  \[
  \leq \sum_i \lambda_i \max_{j: I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{6.26}
  \]
  \[
  = \max_{j: I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{6.27}
  \]
  \[
  = r(A) \tag{6.28}
  \]
If \( x \in P_{\text{ind. set}} \), then

\[
x = \sum \lambda_i 1_{I_i}
\]

for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).

Clearly, for such \( x \), \( x \geq 0 \).

Now, for any \( A \subseteq E \),

\[
x(A) = x^T 1_A = \sum \lambda_i 1_{I_i}^T 1_A
\]

\[
\leq \sum \lambda_i \max_{j: I_j \subseteq A} 1_{I_j}(E)
\]

\[
= \max_{j: I_j \subseteq A} 1_{I_j}(E)
\]

Thus, \( x \in P_r^+ \) and hence \( P_{\text{ind. set}} \subseteq P_r^+ \).
Matroid Polyhedron in 2D

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \]  
(6.29)

- Consider this in two dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \]  
(6.30)

\[ x_1 \leq r(\{v_1\}) \]  
(6.31)

\[ x_2 \leq r(\{v_2\}) \]  
(6.32)

\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \]  
(6.33)
Matroid Polyhedron in 2D

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (6.29) \]

1. Consider this in two dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \quad (6.30) \]
\[ x_1 \leq r(\{v_1\}) \quad (6.31) \]
\[ x_2 \leq r(\{v_2\}) \quad (6.32) \]
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (6.33) \]

2. Because \( r \) is submodular, we have

\[ r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (6.34) \]

so since \( r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\}) \), the last inequality is either touching or active.
Matroid Polyhedron in 2D

\[ x_1 + x_2 = r(\{v_1, v_2\}) = 1 \]
Matroid Polyhedron in 2D

\[ r(\{v_1, v_2\}) = 0 \]
Matroid Polyhedron in 2D

\[ x_1 + x_2 = r(\{v_1, v_2\}) = 2 \]
And, if \( v_2 \) is a loop ...

\[
r(v_1) = 1 \\
r(v_2) = 0 \\n= 1 \]

\[
r(\{v_1, v_2\}) = 1
\]
Matroid Polyhedron in 2D

\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_1 \leq r(\{v_1\}) \]
\[ x_2 \leq r(\{v_2\}) \]
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \]

Possible

Not Possible

Not
Matroid Polyhedron in 3D

\[ P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \]  \hspace{1cm} (6.35)

- Consider this in three dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \]  \hspace{1cm} (6.36)

\[ x_1 \leq r(\{v_1\}) \]  \hspace{1cm} (6.37)

\[ x_2 \leq r(\{v_2\}) \]  \hspace{1cm} (6.38)

\[ x_3 \leq r(\{v_3\}) \]  \hspace{1cm} (6.39)

\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \]  \hspace{1cm} (6.40)

\[ x_2 + x_3 \leq r(\{v_2, v_3\}) \]  \hspace{1cm} (6.41)

\[ x_1 + x_3 \leq r(\{v_1, v_3\}) \]  \hspace{1cm} (6.42)

\[ x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \]  \hspace{1cm} (6.43)
Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.
Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.

So any set of either one or two edges is independent, and has rank equal to cardinality.
Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.

So any set of either one or two edges is independent, and has rank equal to cardinality.

The set of three edges is dependent, and has rank 2.
Matroid Polyhedron in 3D

Two view of $P_r^+$ associated with a matroid
($(e_1, e_2, e_3), \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\})$).
$P_r^+$ associated with the “free” matroid in 3D.
$P_r^+$ associated with the “free” matroid in 3D.
Another Polytope in 3D

Thought question: what kind of polytope might this be?
Another Polytope in 3D

Thought question: what kind of polytope might this be?
Sources for Today’s Lecture

End