Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 10

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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$$f(A) + f(B) - f(A \cup B) + f(A \cap B) \geq -f(A) + 2f(C) - f(B) - f(A) + f(C) + f(B) - f(A \cap B)$$
Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).
Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity

Finals Week: June 9th-13th, 2014.
Theorem 9.2.6

Let $M = (V, \mathcal{I})$ be a matroid, with rank function $r$, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{ w(I) | I \in \mathcal{I} \} = \sum_{i=1}^{n} \lambda_i r(U_i)$$

(9.19)

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i 1_{U_i}$$

(9.20)
Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is
  \[ P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \right\} \]  
  (9.12)
- Now take the rank function $r$ of $M$, and define the following polyhedron:
  \[ P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \]  
  (9.13)

Theorem 9.2.2

\[ P_r^+ = P_{\text{ind. set}} \]  
(9.14)
$P$-basis of $x$ given compact set $P \subseteq \mathbb{R}_+^E$

**Definition 9.2.4 (subvector)**

$y$ is a **subvector** of $x$ if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

**Definition 9.2.5 ($P$-basis)**

Given a compact set $P \subseteq \mathbb{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector $y$ of $x$ is called a **$P$-basis** of $x$ if $y$ maximal in $P$.

In other words, $y$ is a $P$-basis of $x$ if $y$ is a maximal $P$-contained subvector of $x$.

Here, by $y$ being “maximal”, we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon 1_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of $y$ (the properties of $y$ being: in $P$, and a subvector of $x$). In still other words: $y$ is a $P$-basis of $x$ if:

1. $y \leq x$ ($y$ is a subvector of $x$); and
2. $y \in P$ and $y + \epsilon 1_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ ($y$ is maximal $P$-contained).
A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.
  \[
  \text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}|}
  \] (9.25)

- **vector rank**: Given a compact set $P \subseteq \mathbb{R}_+^E$, we can define a form of “vector rank” relative to this $P$ in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to $P$, as:
  \[
  \text{rank}(x) = \max (y(E) : y \leq x, y \in P)
  \] (9.26)

  where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$).

- If $B_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = \max_{y \in B_x} y(E)$.
- If $x \in P$, then $\text{rank}(x) = x(E)$ ($x$ is its own unique self $P$-basis).
- In general, this might be hard to compute and/or have ill-defined properties. We next look at an object that restrains and cultivates this form of rank.
Definition 9.2.4 (polymatroid)

A **polymatroid** is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$.

- Vectors within $P$ (i.e., any $y \in P$) are called **independent**, and any vector outside of $P$ is called **dependent**.
- Since all $P$-bases of $x$ have the same component sum, if $\mathcal{B}_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$. 
Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system \((E, \mathcal{I})\)
2. empty-set containing \(\emptyset \in \mathcal{I}\)
3. down closed, \(\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}\).
4. any maximal set \(I\) in \(\mathcal{I}\), bounded by another set \(A\), has the same matroid rank (any maximal independent subset \(I \subseteq A\) has same size \(|I|\)).

A Polymatroid is:

1. a compact set \(P \subseteq \mathbb{R}^E_+\)
2. zero containing, \(0 \in P\)
3. down monotone, \(0 \leq y \leq x \in P \Rightarrow y \in P\)
4. any maximal vector \(y\) in \(P\), bounded by another vector \(x\), has the same vector rank (any maximal independent subvector \(y \leq x\) has same sum \(y(E)\)).
Polymatroid function and its polyhedron.

**Definition 9.2.4**

A **polymatroid function** is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P_f^+$ associated with a polymatroid function as follows

$$P_f^+ = \{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \} \quad \text{(9.25)}$$

$$= \{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \} \quad \text{(9.26)}$$
Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within $S$ or between $S$ and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

- Observe: $P^+_f$ (at two views):

Which axis is which?
**Associated polyhedron with a polymatroid function**

- Consider modular function $w : V \rightarrow \mathbb{R}_+$ as $w = (1, 1.5, 2)^T$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: $P^+_f$ (at two views):

```
which axis is which?
```

A polymatroid vs. a polymatroid function’s polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function \( f \), its associated polytope is given as
    \[
    P_f^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \} \tag{9.34}
    \]
  - We also have the definition of a polymatroidal polytope (compact subset, zero containing, down-monotone, and \( \forall x \text{ any maximal independent subvector } y \leq x \text{ has same component sum } y(E) \)).

- Is there any relationship between these two polytopes?
- In the next theorem, we show that any \( P_f^+ \)-basis has the same component sum, when \( f \) is a polymatroid function, and \( P_f^+ \) satisfies the other properties so that \( P_f^+ \) is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 9.2.4**

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}_+^E$, and any $P_f^+$-basis $y^x \in \mathbb{R}_+^E$ of $x$, the component sum of $y^x$ is

$$y^x(E) = \text{rank}(x) = \max \left( y(E) : y \leq x, y \in P_f^+ \right)$$

$$= \min \left( x(A) + f(E \setminus A) : A \subseteq E \right)$$

(9.34)

As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in $x$), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max \left\{ y(B) : y \in P_f^+ \right\}$$

(9.35)

In fact, we will ultimately see a number of important consequences of this theorem (other than just that $P_f^+$ is a polymatroid).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P^+_f$ since $f$ is non-negative.
A polymatroid function’s polyhedron is a polymatroid.

**Proof.**

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.

$\mathbf{\text{x}}(\mathbf{A}) \leq \mathbf{y}(\mathbf{A}) \leq f(\mathbf{A}) \quad \forall \mathbf{A}$
Proof.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_E^+$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).
- Goal is to show that any such $y^x$ has $y^x(E) = \text{const}$, dependent only on $x$ and also $f$ (which defines the polytope) but not dependent on $y^x$, the particular $P$-basis.
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.

- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.

- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).

- Goal is to show that any such $y^x$ has $y^x(E) = \text{const}$, dependent only on $x$ and also $f$ (which defines the polytope) but not dependent on $y^x$, the particular $P$-basis.

- Doing so will thus establish that $P_f^+$ is a polymatroid.

...
A polymatroid function’s polyhedron is a polymatroid.

\[ x(E \setminus A) = x(E) - x(A) \]

... proof continued.

First trivial case: could have \( y^x = x \), which happens if 
\( x(A) \leq f(A), \forall A \subseteq E \) (i.e., \( x \in P_f^+ \) strictly). In such case,

\[
\begin{align*}
\min (x(A) + f(E \setminus A) : A \subseteq E) & \quad (9.1) \\
= x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E) & \quad (9.2) \\
= x(E) + \min (f(A) - x(A) : A \subseteq E) & \quad (9.3) \\
= x(E) + \text{D} & \quad (9.4)
\end{align*}
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- 2nd trivial case is when \( x(A) > f(A), \forall A \subseteq E \) (i.e., \( x \notin P_f^+ \) strictly),

\[
y^+(E) = f(E)
\]

\[
y^+(E) = \min (x(A) + f(E \cap A) : A \subseteq E)
\]

\[
y^+(E) = x(E) + \min (f(A) - x(A) : A \subseteq E)
\]

\[
y^+(E) = x(E) + f(E)
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- 2nd trivial case is when $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ strictly),

- Then for any order $(a_1, a_2, \ldots)$ of the elements and $A_i \triangleq (a_1, a_2, \ldots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_{i|A_{i-1}})$, the second inequality by submodularity.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- 2nd trivial case is when $x(A) > f(A)$, $\forall A \subseteq E$ (i.e., $x \not\in P_f^+$ strictly),

- Then for any order $(a_1, a_2, \ldots)$ of the elements and $A_i \triangleq (a_1, a_2, \ldots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1})$, the second inequality by submodularity.

- This gives

$$
\min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(A) - x(A) : A \subseteq E)
$$

(9.5)

$$
= x(E) + \min \left( \sum_i f(a_i | A_{i-1}) - \sum_i x(a_i) : A \subseteq E \right)
$$

(9.6)

$$
= x(E) + f(E) - x(E) = f(E)
$$

(9.7)
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$. 
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by

$$y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (9.9)$$
...proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by

$$y^x(E) = \min \{ x(A) + f(E \setminus A) : A \subseteq E \} \quad (9.9)$$

For any $P_f^+$-basis $y^x$ of $x$, and any $A \subseteq E$, we have that

$$y^x \leq x \quad (9.10)$$

$$y^x(A) \leq x(A) + f(E \setminus A) \quad (9.11)$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$. 

...
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because $y^x \in P^+_f$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by
  \[
y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \tag{9.9}
  \]
- For any $P^+_f$-basis $y^x$ of $x$, and any $A \subseteq E$, we have that
  \[
y^x(E) = y^x(A) + y^x(E \setminus A) \leq x(A) + f(E \setminus A). \tag{9.10}
  \]
  \[
  \leq x(A) + f(E \setminus A). \tag{9.11}
  \]
  *This follows since $y^x \leq x$ and since $y^x \in P^+_f$.*
- Given one $A$ where equality holds, the above min result follows.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C)
\]

which requires equality everywhere above.

Because \( y(B) \leq f(B) \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight. For \( y \in P_f^+ \), it will ultimately use a tight family of tight sets:

\[
D(y) = \{ A : A \subseteq E, y(A) = f(A) \}
\]

...
For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C)$$

(9.12)
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
\begin{align*}
    f(B) + f(C) &= y(B) + y(C) \\
    &= y(B \cap C) + y(B \cup C)
\end{align*}
\]  

\( (9.12) \) 

\( (9.13) \)
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
\begin{align*}
f(B) + f(C) &= y(B) + y(C) \\
&= y(B \cap C) + y(B \cup C) \\
&\leq f(B \cap C) + f(B \cup C)
\end{align*}
\] (9.12)

which requires equality everywhere above.

Because \( y(B) \leq f(B) \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight.

For \( y \in P_f^+ \), it will ultimately use finite this lattice family of tight sets:

\[
D(y) = \{ A : A \subseteq E, y(A) = f(A) \}
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C) \leq f(B) + f(C)
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C) \quad (9.12)$$

$$= y(B \cap C) + y(B \cup C) \quad (9.13)$$

$$\leq f(B \cap C) + f(B \cup C) \quad (9.14)$$

$$\leq f(B) + f(C) \quad (9.15)$$

which requires equality everywhere above.
A polynamatroid function’s polyhedron is a polynamatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C) \leq f(B) + f(C)
\] (9.12-9.15)

which requires equality everywhere above.

Because \( y(B) \leq f(B), \forall B \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C) \leq f(B) + f(C)
\]

which requires equality everywhere above.

- Because \( y(B) \leq f(B), \forall B \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight.

- For \( y \in P_f^+ \), it will be ultimately useful to define this lattice family of tight sets:

\[
D(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \}.
\]
proof continued.

- Also, define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \)
... proof continued.

- Also, define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \}$
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y(E \setminus A) = f(E \setminus A)$).
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \)
- Consider again a \( P_f^+ \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \).
- Let \( E \setminus A = \text{sat}(y^x) \) be the union of all such tight sets (which is also tight, so \( \hat{y}(E \setminus A) = f(E \setminus A) \)).
- Hence, we have

\[
\hat{y}(E) = \hat{y}(A) + \hat{y}(E \setminus A) = x(A) + f(E \setminus A)
\] (9.16)
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, define $\text{sat}(y)$ as $\bigcup \{T : T \in D(y)\}$
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y(E) = y(A) + y(E \setminus A) = x(A) + f(E \setminus A) \quad (9.16)$$

- So we identified the $A$ to be the elements that are non-tight, and achieved the min, as desired.
So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
So, when $f$ is a polymatroid function, $P^+_f$ is a polymatroid.

Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P^+_f$?
A polymatroid is a polymatroid function’s polytope

- So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?

### Theorem 9.3.1

For any polymatroid $P$ (compact subset of $\mathbb{R}^E_+$, zero containing, down-monotone, and $\forall x \in \mathbb{R}^E_+$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \rightarrow \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \}$. 
First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, \; y(A) = f(A) \}$$

(9.17)

Theorem 9.3.2

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.
First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \}$$  \hspace{1cm} (9.17)

**Theorem 9.3.2**

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

**Proof.**

We have already proven this as part of Theorem ??.
First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \}$$  \hspace{1cm} (9.17)

**Theorem 9.3.2**

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

**Proof.**

We have already proven this as part of Theorem ??

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \}$$  \hspace{1cm} (9.18)
Next, a bit on rank\((x)\), join and meet for \(x, y \in \mathbb{R}^E_+\)

- For \(x, y \in \mathbb{R}^E_+\), define vectors \(x \land y \in \mathbb{R}^E_+\) and \(x \lor y \in \mathbb{R}^E_+\) such that, for all \(e \in E\):

\[
(x \lor y)(e) = \max(x(e), y(e))
\]

\[
(x \land y)(e) = \min(x(e), y(e))
\]

(9.19)

(9.20)

Hence,

\[
x \lor y = \left(\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n))\right)
\]

and similarly

\[
x \land y = \left(\min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n))\right)
\]
Next, a bit on rank($x$), join and meet for $x, y \in \mathbb{R}_+^E$

For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

\[ (x \vee y)(e) = \max(x(e), y(e)) \]  
\[ (x \wedge y)(e) = \min(x(e), y(e)) \]

Hence,

\[ x \vee y = (\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n))) \]

and similarly

\[ x \wedge y = (\min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n))) \]

From this, we can define things like an lattices, and other constructs.
Next, a bit on rank$(x)$

- Recall that the matroid rank function is submodular.
Next, a bit on $\text{rank}(x)$

- Recall that the matroid rank function is submodular.
- The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity.
Next, a bit on $\text{rank}(x)$

- Recall that the matroid rank function is submodular.
- The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity.

**Theorem 9.3.3 (vector rank and submodularity)**

Let $P$ be a polymatroid polytope. The vector rank function $\text{rank} : \mathbb{R}_+^E \rightarrow \mathbb{R}$ with $\text{rank}(x) = \max (y(E) : y \leq x, y \in P)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \lor v) + \text{rank}(u \land v) \quad (9.21)$$
Next, a bit on $\text{rank}(x)$

Proof of Theorem 9.3.3.

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$. 

...
Next, a bit on rank($x$)

**Proof of Theorem 9.3.3.**

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that:
  \[ a \leq b \leq u \lor v \]

...
Next, a bit on rank($x$)

Proof of Theorem 9.3.3.

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.

- By the polymatroid property, there exists an independent $b \in P$ such that:
  
  $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.

\[ \text{(9.22)} \]
Next, a bit on \( \text{rank}(x) \)

Proof of Theorem 9.3.3.

- Let \( a \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).

- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: 
  \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \).

- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then then 
  \( a(e) = b(e) \leq \min(u(e), v(e)) \).
Next, a bit on $\text{rank}(x)$

Proof of Theorem 9.3.3.

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.
- Given $e \in E$, if $a(e)$ is maximal due to $P$, then
  $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \land v)(e)$, then
  $a(e) = \min(u(e), v(e)) \leq b(e)$.
Next, a bit on $\text{rank}(x)$

**Proof of Theorem 9.3.3.**

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.
- Given $e \in E$, if $a(e)$ is maximal due to $P$, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  
  If $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.

Therefore, $a = b \land (u \land v)$. 

...
Next, a bit on \( \text{rank}(x) \)

Proof of Theorem 9.3.3.

- Let \( a \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \).
- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then then
  \[
  a(e) = b(e) \leq \min(u(e), v(e)).
  \]
  If \( a(e) \) is maximal due to \((u \land v)(e)\), then
  \[
  a(e) = \min(u(e), v(e)) \leq b(e).
  \]
  Therefore, \( a = b \land (u \land u) \).
- Since \( a = b \land (u \land v) \)
Next, a bit on rank($x$)

Proof of Theorem 9.3.3.

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.
- Given $e \in E$, if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$.
  - If $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  - Therefore, $a = b \land (u \land u)$.
- Since $a = b \land (u \land v)$ and since $b \leq u \lor v$, we get

\[
 a + b
\]

(9.22)

...
Next, a bit on \( \text{rank}(x) \)

**Proof of Theorem 9.3.3.**

- Let \( a \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).

- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
  \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \).

- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then then
  \[ a(e) = b(e) \leq \min(u(e), v(e)) \].
  If \( a(e) \) is maximal due to \( (u \land v)(e) \), then
  \[ a(e) = \min(u(e), v(e)) \leq b(e) \].
  Therefore, \( a = b \land (u \land u) \).

- Since \( a = b \land (u \land v) \) and since \( b \leq u \lor v \), we get
  \[ a + b = b + b \land u \land v \] (9.22)

\[ \cdots \]
Next, a bit on rank($x$)

**Proof of Theorem 9.3.3.**

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.

- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.

- Given $e \in E$, if $a(e)$ is maximal due to $P$, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \land (u \land u)$.

- Since $a = b \land (u \land v)$ and since $b \leq u \lor v$, we get
  \[ a + b = b + b \land u \land v = b \land u + b \land v \]  \hspace{1cm} (9.22)
Next, a bit on $\text{rank}(x)$

Proof of Theorem 9.3.3.

- Let $a$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$.
- Given $e \in E$, if $a(e)$ is maximal due to $P$, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \land (u \land u)$.
- Since $a = b \land (u \land v)$ and since $b \leq u \lor v$, we get
  \[ a + b = b + b \land u \lor v = b \land u + b \land v \]  (9.22)

To see this, consider each case where either $b$ is the minimum, or $u$ is minimum with $b \leq v$, or $v$ is minimum with $b \leq u$. ...
Next, a bit on $\text{rank}(x)$

... proof of Theorem 9.3.3.

But $b \wedge u$ and $b \wedge v$ are independent subvectors of $u$ and $v$ respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$. 
Next, a bit on $\text{rank}(x)$

... proof of Theorem 9.3.3.

- But $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.

- Hence,
  $$\text{rank}(u \land v) + \text{rank}(u \lor v)$$
Next, a bit on $\text{rank}(x)$

... proof of Theorem 9.3.3.

- But $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.

- Hence,

\[
\text{rank}(u \land v) + \text{rank}(u \lor v) = a(E) + b(E) \tag{9.23}
\]
Next, a bit on $\text{rank}(x)$

\[ \text{proof of Theorem 9.3.3.} \]

- But $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.
- Hence,
  \[
  \text{rank}(u \land v) + \text{rank}(u \lor v) = a(E) + b(E) \tag{9.23}
  \]
  \[
  = (b \land u)(E) + (b \land v)(E) \tag{9.24}
  \]
Next, a bit on \( \text{rank}(x) \)

... proof of Theorem 9.3.3.

But \( b \land u \) and \( b \land v \) are independent subvectors of \( u \) and \( v \) respectively, so \( (b \land u)(E) \leq \text{rank}(u) \) and \( (b \land v)(E) \leq \text{rank}(v) \).

Hence,

\[
\text{rank}(u \land v) + \text{rank}(u \lor v) = a(E) + b(E) = (b \land u)(E) + (b \land v)(E) \leq \text{rank}(u) + \text{rank}(v)
\]

(9.23)  
(9.24)  
(9.25)
Note the remarkable similarity between the proof of Theorem 9.3.3 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.
Note the remarkable similarity between the proof of Theorem 9.3.3 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 9.3.1, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P_f^+$. 
Note the remarkable similarity between the proof of Theorem 9.3.3 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 9.3.1, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P_f^+$. Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper “Submodular Functions, Matroids, and Certain Polyhedra”).
Proof of Theorem 9.3.1.

- We are given a polymatroid $P$. 

**Proof of Theorem 9.3.1.**
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- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty and $\alpha_{\text{max}} = \text{rank}(\infty 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$. 

...
Proof of Theorem 9.3.1.

- We are given a polymatroid \( P \).

- Define \( \alpha_{\max} \triangleq \max \{ x(E) : x \in P \} \), and note that \( \alpha_{\max} > 0 \) when \( P \) is non-empty, and \( \alpha_{\max} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\max} \mathbf{1}_E) \).

- Hence, for any \( x \in P \), \( x(e) \leq \alpha_{\max}, \forall e \in E \).
Proof of Theorem 9.3.1.

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}$, $\forall e \in E$.
- Define a function $f : 2^V \rightarrow \mathbb{R}$ as, for any $A \subseteq E$,

\[ f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \] (9.26)
Proof of Theorem 9.3.1.

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  $$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \quad (9.26)$$
  - Then $f$ is submodular since
  $$f(A) + f(B)$$
Proof of Theorem 9.3.1.

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$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B) \quad (9.27)$$
Proof of Theorem 9.3.1

We are given a polymatroid $P$.

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Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B) \quad (9.27)$$

$$\geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B) \quad (9.28)$$
Proof of Theorem 9.3.1.

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}$, $\forall e \in E$.
- Define a function $f : 2^V \rightarrow \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \quad (9.26)$$

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$$\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \lor \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \land \alpha_{\text{max}} \mathbf{1}_B) \quad (9.28)$$

$$= \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cup B}) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cap B}) \quad (9.29)$$
Proof of Theorem 9.3.1.

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- Define $\alpha_{\max} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\max} > 0$ when $P$ is non-empty, and $\alpha_{\max} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\max} \mathbf{1}_E)$.
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- Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\max} \mathbf{1}_A) + \text{rank}(\alpha_{\max} \mathbf{1}_B) \quad (9.27)$$

$$\geq \text{rank}(\alpha_{\max} \mathbf{1}_A \lor \alpha_{\max} \mathbf{1}_B) + \text{rank}(\alpha_{\max} \mathbf{1}_A \land \alpha_{\max} \mathbf{1}_B) \quad (9.28)$$

$$= \text{rank}(\alpha_{\max} \mathbf{1}_{A \cup B}) + \text{rank}(\alpha_{\max} \mathbf{1}_{A \cap B}) \quad (9.29)$$

$$= f(A \cup B) + f(A \cap B) \quad (9.30)$$
Proof of Theorem 9.3.1.

- Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
Proof of Theorem 9.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.
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Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P_f^+$ defined as:

$$P_f^+ = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \}$$

(9.31)
Proof of Theorem 9.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P_f^+$ defined as:

$$P_f^+ = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \} \quad (9.31)$$

Given an $x \in P$, then for any $A \subseteq E$,

$$x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\max} \mathbf{1}_A \} = \text{rank}(\alpha_{\max} \mathbf{1}_A) = f(A),$$

$$x(e) \leq \max P_f^+ \quad \forall e \in E.$$
Proof of Theorem 9.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P^+_f$ defined as:

$$P^+_f = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \}$$  \hspace{1cm} (9.31)

Given an $x \in P$, then for any $A \subseteq E$,

$$x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\text{max}} \mathbf{1}_A \} = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) = f(A),$$

therefore $x \in P^+_f$. 

...
Proof of Theorem 9.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P_f^+$ defined as:

\[
P_f^+ = \left\{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \right\}
\]  

(9.31)

Given an $x \in P$, then for any $A \subseteq E$,

\[
x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\max} 1_A \} = \text{rank}(\alpha_{\max} 1_A) = f(A),
\]

therefore $x \in P_f^+$.

Hence, $P \subseteq P_f^+$.
Proof of Theorem 9.3.1.

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Consider the polytope $P_f^+$ defined as:

$$P_f^+ = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \} \quad (9.31)$$

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therefore $x \in P_f^+$.

Hence, $P \subseteq P_f^+$.

We will next show that $P_f^+ \subseteq P$ to complete the proof.

...
Proof of Theorem 9.3.1.

Let $x \in P^+_f$ be chosen arbitrarily (goal is to show that $x \in P$).
Proof of Theorem 9.3.1.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. 

...
Proof of Theorem 9.3.1.

Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).

Suppose \( x \notin P \). Then, choose \( y \) to be a \( P \)-basis of \( x \) that maximizes the number of \( y \) elements strictly less than the corresponding \( x \) element. I.e., that maximizes \( |N(y)| \), where

\[
N(y) = \{ e \in E : y(e) < x(e) \}
\]

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Proof of Theorem 9.3.1.

Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).

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\]

Choose \( w \) between \( y \) and \( x \), so that

\[
y \leq w \triangleq (y + x)/2 \leq x \tag{9.33}
\]

so \( y \) is also a \( P \)-basis of \( w \).
Proof of Theorem 9.3.1.

- Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).
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y \leq w \triangleq \frac{y + x}{2} \leq x \tag{9.33}
\]

so \( y \) is also a \( P \)-basis of \( w \).

- Hence, \( \text{rank}(x) = \text{rank}(w) \), and the set of \( P \)-bases of \( w \) are also \( P \)-bases of \( x \).
Proof of Theorem 9.3.1.

For any \( A \subseteq E \), define \( x_A \in \mathbb{R}^E_+ \) as

\[
x_A(e) = \begin{cases} 
 x(e) & \text{if } e \in A \\
 0 & \text{else} 
\end{cases}
\]  

(9.34)

*note this is an analogous definition to \( 1_A \) but for a non-unity vector.*
Proof of Theorem 9.3.1.

For any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

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Note this is an analogous definition to $1_A$ but for a non-unity vector.

Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\text{max}}1_{N(y)}) \quad (9.35)$$

the last inequality follows since $w \leq x \in P_f^+$, and $y \leq w$. 
Proof of Theorem 9.3.1.

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the last inequality follows since $w \leq x \in P_f^+$, and $y \leq w$.

- Thus, $y \land x_{N(y)}$ is not a $P$-basis of $w \land x_{N(y)}$ since, over $N(y)$, it is neither tight at $w$ nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).
Proof of Theorem 9.3.1.

- We can extend $y \land x_{N(y)}$ to be a $P$-basis of $w \land x_{N(y)}$ since $y \land x_{N(y)} < w \land x_{N(y)}$.

This contradiction means that we must have had $x \in P$. Therefore, $P + f = P$. 

$\blacksquare$
Proof of Theorem 9.3.1.

- We can extend $y \wedge x_N(y)$ to be a $P$-basis of $w \wedge x_N(y)$ since $y \wedge x_N(y) \prec w \wedge x_N(y)$.
- This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \& x$. 

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- Now, we have $\hat{y}(N(y)) > y(N(y))$,
Proof of Theorem 9.3.1.

- We can extend \( y \land x_{N(y)} \) to be a \( P \)-basis of \( w \land x_{N(y)} \) since 
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- This \( P \)-basis, in turn, can be extended to be a \( P \)-basis \( \hat{y} \) of \( w \land x \).
- Now, we have \( \hat{y}(N(y)) > y(N(y)) \),
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Proof of Theorem 9.3.1.

- We can extend \( y \land x_{N(y)} \) to be a \( P \)-basis of \( w \land x_{N(y)} \) since 
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- hence \( \hat{y}(e) < y(e) \) for some \( e \notin N(y) \).
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- Thus, \( \hat{y} \) is a base of \( x \), which violates the maximality of \( |N(y)| \).
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- This contradiction means that we must have had \( x \in P \).
Proof of Theorem 9.3.1.

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- Thus, \( \hat{y} \) is a base of \( x \), which violates the maximality of \( |N(y)| \).
- This contradiction means that we must have had \( x \in P \).
- Therefore, \( P_f^+ = P \).
More on polymatroids

Theorem 9.3.4

A polymatroid can equivalently be defined as a pair \((E, P)\) where \(E\) is a finite ground set and \(P \subseteq R^E_+\) is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)
More on polymatroids

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1. every subvector of an independent vector is independent (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)

2. If \(u, v \in P\) (i.e., are independent) and \(u(E) < v(E)\), then there exists a vector \(w \in P\) such that

\[
u < w \leq u \lor v\]

(9.36)
More on polymatroids

**Theorem 9.3.4**

A polymatroid can equivalently be defined as a pair $(E, P)$ where $E$ is a finite ground set and $P \subseteq \mathbb{R}_+^E$ is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)
2. If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that

$$ u < w \leq u \lor v \quad (9.36) $$

---

**Corollary 9.3.5**

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$. 

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Prof. Jeff Bilmes  
EE596b/Spring 2014/Submodularity - Lecture 9 - April 28th, 2014  
F34/66 (pg.107/231)
The next slide comes from lecture 5.
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 9.3.1 (Matroid (by bases))

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
More on polymatroids

For any compact set $P$, $b$ is a base of $P$ if it is a maximal subvector within $P$. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

Theorem 9.3.6

A polymatroid can equivalently be defined as a pair $(E, P)$ where $E$ is a finite ground set and $P \subseteq \mathbb{R}^E_+$ is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)
2. if $b, c$ are bases of $P$ and $d$ is such that $b \wedge c < d < b$, then there exists an $f$, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of $P$
3. All of the bases of $P$ have the same rank.

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).
basic
also, a word on terminology

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
also, a word on terminology

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\),
also, a word on terminology

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\),
- But now we see that \((E, f)\) is equivalent to a polymatroid polytope, so this is sensible.
Where are we going with this?

Consider the right hand side of Theorem ??:

$$\min (x(A) + f(E \setminus A) : A \subseteq E)$$
Where are we going with this?

- Consider the right hand side of Theorem ??:
  \[
  \min (x(A) + f(E \setminus A) : A \subseteq E)
  \]

- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
Consider the right hand side of Theorem 2:
\[ \min (x(A) + f(E \setminus A) : A \subseteq E) \]

We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

As a bit of a hint on what’s to come, note that we can write it as:
\[ x(E) + \min (f(A) - x(A) : A \subseteq E) \] where \( f \) is a polymatroid function.
Another Interesting Fact: Matroids from polymatroid functions

**Theorem 9.3.7**

Given integral polymatroid function \( f \), let \( (E, F) \) be a set system with ground set \( E \) and set of subsets \( F \) such that

\[
\forall F \in F, \quad \forall \emptyset \subset S \subseteq F, \quad |S| \leq f(S)
\]  

(9.37)

Then \( M = (E, F) \) is a matroid.

**Proof.**

Exercise

And its rank function is Exercise.
Considering Theorem ??, the matroid case is now a special case, where we have that:

**Corollary 9.3.8**

We have that:

$$\max \{ y(E) : y \in \text{ind. set}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}$$

(9.38)

where $r_M$ is the matroid rank function of some matroid.
Most violated inequality problem in matroid polytope case

Consider

\[ P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \]  

(9.39)
Most violated inequality problem in matroid polytope case

- Consider

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \} \]  

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- We saw before that \( P_r^+ = P_{\text{ind. set}} \).
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- Consider

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- We saw before that \( P_r^+ = P_{\text{ind. set}} \).

- Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \not\in P_r^+ \).
Most violated inequality problem in matroid polytope case

Consider

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \} \quad (9.39) \]

We saw before that \( P_r^+ = P_{\text{ind. set}}. \)

Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \notin P_r^+. \)

The most violated inequality when \( x \) is considered w.r.t. \( P_r^+ \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A), \) i.e.,

\[ \max \{ x(A) - r_M(A) : A \subseteq E \}. \]
Consider

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \} \quad (9.39) \]

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Suppose we have any \( x \in \mathbb{R}^E_+ \) such that \( x \notin P_r^+ . \)

The most violated inequality when \( x \) is considered w.r.t. \( P_r^+ \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e.,

\[ \max \{ x(A) - r_M(A) : A \subseteq E \}. \]

This corresponds to \( \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \).
Most violated inequality problem in matroid polytope case

- Consider

\[ P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \right\} \quad (9.39) \]

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- The most violated inequality when \( x \) is considered w.r.t. \( P_r^+ \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e.,
  \[ \max \left\{ x(A) - r_M(A) : A \subseteq E \right\}. \]
- This corresponds to \( \min \left\{ r_M(A) + x(E \setminus A) : A \subseteq E \right\} \) since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \).
- More importantly, \( \min \left\{ r_M(A) + x(E \setminus A) : A \subseteq E \right\} \) a form of submodular function minimization, namely
  \[ \min \left\{ r_M(A) - x(A) : A \subseteq E \right\} \] for a submodular function consisting of a difference of matroid rank and modular (so no longer necessarily monotone, nor positive).
**Problem to Solve**

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}_+^E$;
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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. Of course, for any such $y$ we must have that $y(E) \leq r(A) + x(E \setminus A)$. 

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- By the above theorem, the existence of such an \( A \) will certify that \( y(E) \) is maximal in \( P_{\text{ind. set}} \), \( A \) is minimal in terms of \( f(A) \overset{\text{def}}{=} r_M(A) - x(A) \) (thus most violated).
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- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}^E_+$;

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- This can also be used to test membership in $\text{P}_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of $f$ at $A$. 
Problem to Solve

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- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathbb{R}^E_+$;

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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of $f$ at $A$.

- This will also run in polynomial time.
Idea of the algorithm

- We build up $y$ from the ground up.
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- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{I_i}$. 
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- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
Idea of the algorithm

- We build up $y$ from the ground up.
- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i 1_{I_i}$.
- We gradually build up $y$ by adding new independent sets (and augmenting $J$), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we’ll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we’ll keep $y \leq x$. 
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- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we’ll keep $y \leq x$.
- It’s going to take us a few lectures to fully develop this algorithm, so please keep in mind of the overall goal.
Consider a bipartite graph $G = (V, F, E)$ where left nodes are $V$, right nodes are $F$, and $E \subseteq V \times F$ are the only edges.
Bipartite Matching

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A matching $A \subseteq E$ is a subset of edges such that no two edges are incident to the same vertex.
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- A node $j$ is matched in $A$ if $(j, k) \in A$ for some $k \in F$, and otherwise $j$ is called unmatched. Likewise for some $k \in F$. 
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- Given $A \subseteq E$, an alternating path $S$ (relative to $A$) is an (undirected) path of unique edges that are alternatively in $A$ and not in $A$. I.e., if $S = (e_1, e_2, \ldots, e_s)$ is an alternating path, then $S_{1/2} \overset{\text{def}}{=} S \setminus A$ where $S_{1/2}$ is either the odd or the even elements of $S$. 
Consider a bipartite graph $G = (V, F, E)$ where left nodes are $V$, right nodes are $F$, and $E \subseteq V \times F$ are the only edges.

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An $A \subseteq E$ is an **augmenting path** if it is an alternating path between two unmatched vertices.
Bipartite Matching

- Given a matching $A \subseteq E$ (which might be empty), we can increase the matching if we can find an augmenting path $S$. 

\[
A' = A \setminus S \cup S \setminus A
\]

The algorithm becomes:

**Algorithm 8.1: Alternating Path Bipartite Matching**

1. Let $A$ be an arbitrary (including empty) matching in $G = (V,F,E)$;
2. while There exists an augmenting path $S$ in $G$ do
3. $A \leftarrow A \setminus S$;
4. This can easily be made to run in $O(m^2n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made faster as well (see Schrijver-2003).
### Bipartite Matching

- Given a matching \( A \subseteq E \) (which might be empty), we can increase the matching if we can find an augmenting path \( S \).
- The updated matching becomes \( A' = A \setminus S \cup S \setminus A = A \oplus S \), where \( \oplus \) is the symmetric difference operator.
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**Algorithm 8.1**: Alternating Path Bipartite Matching

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**Algorithm 8.1: Alternating Path Bipartite Matching**

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- This can easily be made to run in $O(m^2n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).
Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.
Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.
Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.

![Bipartite Matching Example Diagram]
Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.
Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).
Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.
At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible). At this point, matching is maximum cardinality.
The next slide is from lecture 7 and the one after from lecture 4.
Matroid Intersection

Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

**Theorem 9.5.5**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$
(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right)
$$

This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$$
(f_1 * f_2)(Y) = \min_{X \subseteq Y} \left( f_1(X) + f_2(Y \setminus X) \right)
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Let $V$ be our ground set.
Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_{\ell}$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

$$I = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}.$$  \hspace{1cm} (9.3)

where $k_1, \ldots, k_{\ell}$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.
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- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$. 

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- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

- We’ll show that property (I3’) in Def ?? holds. If $X, Y \in I$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
Why might we want to do matroid intersection?
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$I \in \mathcal{I}_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$ and $I \in \mathcal{I}_F$ if $|I \cap (v, F)| \leq 1$ for all $v \in V$. 
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Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
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Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.

For the bipartite graph case, therefore, this can be solved in polynomial time.
Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on the same underlying edges.
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Consider two cycle matroids associated with these graphs $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. They might be very different (e.g., an edge might be between two distinct nodes in $G_1$ but the same edge is a loop in multi-graph $G_2$.)
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We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either $M_1$, $M_2$, or both).
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We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either $M_1$, $M_2$, or both).

This is again a matroid intersection problem.
Matroid Intersection and TSP

- Definition: a **Hamiltonian cycle** is a cycle that passes through each node exactly once.
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Given directed graph $G$, goal is to find such a Hamiltonian cycle.
Matroid Intersection and TSP

- Definition: a Hamiltonian cycle is a cycle that passes through each node exactly once.
- Given directed graph $G$, goal is to find such a Hamiltonian cycle.
- From $G$ with $n$ nodes, create $G'$ with $n + 1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to $v_1^+, v_1^-$, and have all outgoing edges from $v_1$ come instead from $v_1^+$ and all edges incoming to $v_1$ go instead to $v_1^-$. 
Matroid Intersection and TSP

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- Let $M_1$ be the cycle matroid on $G'$. 
Definition: a **Hamiltonian cycle** is a cycle that passes through each node exactly once.

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Let $M_1$ be the cycle matroid on $G'$.

Let $M_2$ be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$. 

Let $M_3$ be the partition matroid having as independent sets those that have no more than one edge entering any node — i.e., $I \in \mathcal{I}(M_3)$ if $|I \cap \delta^-(v)| \leq 1$ for all $v \in V(G')$. 

Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1, M_2, \text{and } M_3$. 

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Prof. Jeff Bilmes

EE596b/Spring 2014/Submodularity - Lecture 9 - April 28th, 2014
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- Let $M_1$ be the cycle matroid on $G'$.
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- Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1$, $M_2$, and $M_3$.  

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But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...
A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 9.5.1**

**Matroid (by circuits)** Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.

1. $(C1)$ $C$ is the collection of circuits of a matroid;
2. $(C2)$ if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. $(C3)$ if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y;$
Lemma 9.5.2

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in $M$.

Proof.

Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$. 


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- Suppose, to the contrary, that there are two distinct circuits \( C_1, C_2 \) such that \( C_1 \cup C_2 \subseteq I \cup \{e\} \).
- Then \( e \in C_1 \cap C_2 \), and by (C2), there is a circuit \( C_3 \) of \( M \) s.t. \( C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I \).
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- This contradicts the independence of $I$. 


Circuits

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- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit $C_3$ of $M$ s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$.

- This contradicts the independence of $I$.

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in $M$ w.r.t. $I$ and $e$).
Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \text{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$. 
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- If $I + v_1 \in \mathcal{I}_2$, then $v_1$ is “augmenting”, and we can augment $I$ to $I + v_1$ and still be independent in both $M_1$ and $M_2$. 
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- If $I + v_1 \notin \mathcal{I}_2$, then $\exists C_2(I, v_1)$ a circuit in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}_2(I + v_1 - v_2)$) is again independent in $M_2$.
- $I + v_1 - v_2$ is also independent in $M_1$. 
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- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
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- If $I + v_1 \notin \mathcal{I}_2$, then there exists a circuit $C_2(I, v_1)$ in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ such that $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}_2(I + v_1 - v_2)$) is again independent in $M_2$. $I + v_1 - v_2$ is also independent in $M_1$.
- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$. 
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \operatorname{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$.
- If $I + v_1 \in \mathcal{I}_2$, then $v_1$ is “augmenting”, and we can augment $I$ to $I + v_1$ and still be independent in both $M_1$ and $M_2$.
- If $I + v_1 \notin \mathcal{I}_2$, $\exists C_2(I, v_1)$ a circuit in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\operatorname{span}_2(I) = \operatorname{span}_2(I + v_1 - v_2)$) is again independent in $M_2$. $I + v_1 - v_2$ is also independent in $M_1$.
- Next choose a $v_3 \in \operatorname{span}_1(I) - \operatorname{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
- Then $\operatorname{span}_1(I) = \operatorname{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \operatorname{span}_1(I)$, so $\operatorname{span}_1(I + v_1) = \operatorname{span}_1(I + v_1 - v_2 + v_3)$.
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \text{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$.
- If $I + v_1 \in \mathcal{I}_2$, then $v_1$ is “augmenting”, and we can augment $I$ to $I + v_1$ and still be independent in both $M_1$ and $M_2$.
- If $I + v_1 \notin \mathcal{I}_2$, $\exists C_2(I, v_1)$ a circuit in $M_2$, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}_2(I + v_1 - v_2)$) is again independent in $M_2$. $I + v_1 - v_2$ is also independent in $M_1$.
- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed $v_2$ from it.
- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.
- But $I + v_1 - v_2 + v_3$ might not be independent in $M_2$ again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.
Matroid Intersection Algorithm Idea

- Hopefully (eventually) we’ll find an odd length sequence $S = (v_1, v_2, \ldots, v_s)$ such that we will be independent in both $M_1$ and $M_2$ and thus be one greater in size than $I$. 

We then replace $I$ with $I \setminus S$ (quite analogous to the bipartite matching case), and start again.
Matroid Intersection Algorithm Idea

- Hopefully (eventually) we’ll find an odd length sequence \( S = (v_1, v_2, \ldots, v_s) \) such that we will be independent in both \( M_1 \) and \( M_2 \) and thus be one greater in size than \( I \).
- We then replace \( I \) with \( I \ominus S \) (quite analogous to the bipartite matching case), and start again.
Graphic Matroid Intersection Example

Consider the following two graphs $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. Any edge is independent in both (an augmenting “sequence”) since a single edge can’t create a circuit starting at $I = \emptyset$. We start with $e_4$. 
Adding edge $I \leftarrow I + e_4$ creates a circuit neither in $M_1$ nor $M_2$. We can add another single edge w/o creating a circuit in either matroid.
$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we’re still independent in $M_2$, but no further single edge additions possible w/o creating a circuit (why?).

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) at (0,0) [circle,draw] {$e_2$};
\node (B) at (1,1) [circle,draw] {$e_4$};
\node (C) at (2,0) [circle,draw] {$e_3$};
\node (D) at (1,-1) [circle,draw] {$e_6$};
\node (E) at (0,1) [circle,draw] {$e_1$};
\node (F) at (2,1) [circle,draw] {$e_8$};
\node (G) at (2,2) [circle,draw] {};\node (H) at (0,2) [circle,draw] {};
\draw[red, thick] (A) -- (B);
\draw[red, thick] (B) -- (C);
\draw[red, thick] (C) -- (D);
\draw[red, thick] (D) -- (E);
\draw[red, thick] (E) -- (F);
\draw[red, thick] (F) -- (G);
\draw[red, thick] (G) -- (H);
\draw[red, thick] (H) -- (A);
\end{tikzpicture}
\end{figure}

\begin{figure}
\centering
\begin{tikzpicture}
\node (A) at (0,0) [circle,draw] {$e_2$};
\node (B) at (1,1) [circle,draw] {$e_5$};
\node (C) at (2,0) [circle,draw] {$e_3$};
\node (D) at (1,-1) [circle,draw] {$e_6$};
\node (E) at (0,1) [circle,draw] {$e_1$};
\node (F) at (2,1) [circle,draw] {$e_8$};
\node (G) at (2,2) [circle,draw] {};\node (H) at (0,2) [circle,draw] {};
\draw[red, thick] (A) -- (B);
\draw[red, thick] (B) -- (C);
\draw[red, thick] (C) -- (D);
\draw[red, thick] (D) -- (E);
\draw[red, thick] (E) -- (F);
\draw[red, thick] (F) -- (G);
\draw[red, thick] (G) -- (H);
\draw[red, thick] (H) -- (A);
\end{tikzpicture}
\end{figure}
$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we’re still independent in $M_2$, but no further single edge additions possible w/o creating a circuit (why?). We need a multi-edge “augmenting sequence” if it exists.
Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in $I$, green is new). We choose $e_2 \in E - \text{span}_1(I)$, but now $I + e_2$ is not independent in $M_2$. 
So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.
Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add.
Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add. In this case, we not only have $\text{span}_1(I + e_2) = \text{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$. 
Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$. 

G1

\begin{itemize}
  \item $e_2$
  \item $e_5$
  \item $e_1$
  \item $e_7$
  \item $e_8$
\end{itemize}

G2

\begin{itemize}
  \item $e_1$
  \item $e_3$
  \item $e_5$
  \item $e_2$
  \item $e_4$
  \item $e_6$
  \item $e_7$
  \item $e_8$
\end{itemize}
At this point, are any further augmenting sequences possible? Exercise.
Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).
Alternating and Augmenting Sequences

- Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

- Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

  1. For all even $i$, $\text{span}_2(I \setminus S_i) = \text{span}_2(I)$ which implies that $I \setminus S_i \in \mathcal{I}_2$.

  2. For all odd $i$, $\text{span}_1(I \setminus S_i) = \text{span}_1(I + e_1)$, and therefore $I \setminus S_i \in \mathcal{I}_1$.

Lastly, if also, $|S| = s$ is odd, and $I \setminus S \in \mathcal{I}_2$, then $S$ is called an augmenting sequence w.r.t. $I$. 
Alternating and Augmenting Sequences

Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in \mathcal{I}_1$

Alternating and Augmenting Sequences

Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in \mathcal{I}_1$
2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.
Alternating and Augmenting Sequences

- Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

- Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

1. $I + e_1 \in \mathcal{I}_1$
2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.
3. For all odd $i$, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

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3. For all odd $i$, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.

Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then $S$ is called an augmenting sequence w.r.t. $I$. 
If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$.
If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$.

Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover (and this next thing should be a theorem), if there is an augmenting sequence, then the intersection is not maximum.
If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$.

Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover (and this next thing should be a theorem), if there is an augmenting sequence, then the intersection is not maximum.

We next wish to show that, if the intersection is not maximum, then there is an augmenting sequence.
Border graphs

We construct an auxiliary directed bipartite graph (Border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$. 
Border graphs

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- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $e_i \leftarrow e_j$ directed edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$. Note $e_j \in I$ and $e_i \in E \setminus I$. 

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Border graphs

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- If $e_i \notin \text{span}_1(I)$, then $e_i$ has in-degree zero (a source).
Border graphs

- We construct an auxiliary directed bipartite graph \((\text{Border graph})\) \(B(I) = (E \setminus I, I, Z)\), relative to the current \(I\), that will help us with this problem. The graph has only directed edges from \(E \setminus I\) to \(I\), or from \(I\) back to \(E \setminus I\).

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- If \(e_i \notin \text{span}_1(I)\), then \(e_i\) has in-degree zero (a source).

- Right-going edges: For each \(e_i \in \text{span}_2(I) \setminus I\), create \(e_i \rightarrow e_j\) edge \((e_i, e_j) \in Z\) for any \(e_j \in C_2(I, e_i) \setminus \{e_i\}\).
Border graphs

- We construct an auxiliary directed bipartite graph (Border graph) \( B(I) = (E \setminus I, I, Z) \), relative to the current \( I \), that will help us with this problem. The graph has only directed edges from \( E \setminus I \) to \( I \), or from \( I \) back to \( E \setminus I \).

- Left-going edges: For each \( e_i \in \text{span}_1(I) \setminus I \), create \( e_i \leftarrow e_j \) directed edge \((e_j, e_i) \in Z\) for any \( e_j \in C_1(I, e_i) \setminus \{e_i\} \). Note \( e_j \in I \) and \( e_i \in E \setminus I \).

- If \( e_i \notin \text{span}_1(I) \), then \( e_i \) has in-degree zero (a source).

- Right-going edges: For each \( e_i \in \text{span}_2(I) \setminus I \), create \( e_i \rightarrow e_j \) edge \((e_i, e_j) \in Z\) for any \( e_j \in C_2(I, e_i) \setminus \{e_i\} \).

- If \( e_i \notin \text{span}_2(I) \), then \( e_i \) has out-degree zero (a sink).
\{e_2, e_7, e_8\} are sources and \{e_1, e_3, e_6\} are sinks.
\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\} and \text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}
Border graph Example

\{e_2, e_7, e_8\} are sources and \{e_1, e_3, e_6\} are sinks.
\[\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\} \quad \text{and} \quad \text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}\]

Augmenting sequences are \((e_2, e_4, e_1), (e_2, e_4, e_3), \text{ and } (e_2, e_4, e_6)\), all of which are dipaths in the Border graph.
Border graph Example

{e_2, e_7, e_8} are sources and {e_1, e_3, e_6} are sinks.
\[ \text{span}_1(I) \setminus I = \{e_1, e_3, e_6\} \text{ and } \text{span}_2(I) \setminus I = \{e_7, e_2, e_8\} \]

Augmenting sequences are \((e_2, e_4, e_1), (e_2, e_4, e_3), \) and \((e_2, e_4, e_6),\)
all of which are dipaths in the Border graph.

Are there others?
Lemma 9.5.3

If $S$ is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then $S$ is an augmenting sequence w.r.t. $I$. 
Identifying Augmenting Sequences

Lemma 9.5.3

If $S$ is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then $S$ is an augmenting sequence w.r.t. $I$.

Lemma 9.5.4

Let $I$ and $J$ be intersections such that $|I| + 1 = |J|$. Then there exists a source-sink path $S$ in $B(I)$ where $S \subseteq I \oplus J$.
Theorem 9.5.5

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. $I_p$. 
Identifying Augmenting Sequences

**Theorem 9.5.5**

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p + 1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. $I_p$.

**Theorem 9.5.6**

An intersection is of maximum cardinality iff it admits no augmenting sequence.
**Identifying Augmenting Sequences**

**Theorem 9.5.5**

Let $I_p$ and $I_{p+1}$ be intersections of $M_1$ and $M_2$ with $p$ and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. $I_p$.

**Theorem 9.5.6**

An intersection is of maximum cardinality iff it admits no augmenting sequence.

**Theorem 9.5.7**

For any intersection $I$, there exists a maximum cardinality intersection $I^*$ such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$. 
Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$. 
Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$.

We wish to, if possible, partition $E$ into $k$ blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$. 
Matroid Partition Problem

- Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have $k$ of them on the same ground set $E$.

- We wish to, if possible, partition $E$ into $k$ blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$.

- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.
Theorem 9.6.1

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$ (9.40)

where $r_i$ is the rank function of $M_i$. 

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Matroid Partition Problem

Theorem 9.6.1

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$

(9.40)

where $r_i$ is the rank function of $M_i$.

- Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

$$|A| \leq kr(A) \ \forall A \subseteq E$$

(9.41)
Theorem 9.6.1

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $I \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in I_i$ is independent in matroid $i$, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$

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where $r_i$ is the rank function of $M_i$.

- Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

$$|A| \leq kr(A) \ \forall A \subseteq E$$

(9.41)

- But considering vector of all ones $1 \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k} 1(A) \leq r(A) \ \forall A \subseteq E$$

(9.42)
Recall definition of matroid polytope

\[ P_r^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \} \]  

(9.43)
Matroid Partition Problem

- Recall definition of matroid polytope

\[ P_r^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \} \]  

(9.43)

- Then we see that this special case of the matroid partition problem is just testing if \( \frac{1}{k} \mathbf{1} \in P_r^+ \), a problem of testing the membership in matroid polyhedra.