\[
\begin{align*}
 f(A) + f(B) &\geq f(A \cup B) + f(A \cap B) \\
 &\geq f(A) + f(B)
\end{align*}
\]
Logistics

Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, exchange capacity, minimum norm point algorithm and the lattice of minimizers of a submodular function, Lovasz extension
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.
**Tight sets** $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, y(A) = f(A) \} \quad (13.18)$$

### Theorem 13.2.1

*For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.*

### Proof.

We have already proven this as part of Theorem ??.

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \quad (13.19)$$

---

**Fundamental circuits in matroids**

### Lemma 13.2.3

*Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{ e \}$ contains at most one circuit in $M$.*

### Proof.

- Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{ e \}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit $C_3$ of $M$ s.t.
  $$C_3 \subseteq (C_1 \cup C_2) \setminus \{ e \} \subseteq I$$
- This contradicts the independence of $I$.

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{ e \}$ (commonly called the fundamental circuit in $M$ w.r.t. $I$ and $e$).
Matroid Partition Problem

**Theorem 13.2.1**

Let $M_i$ be a collection of $k$ matroids as described. Then, a set $S \subseteq E$ can be partitioned into $k$ subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid $i$, if and only if, for all $A \subseteq S$,

$$|A| \leq \sum_{i=1}^{k} r_i(A)$$  \hspace{1cm} (13.1)

where $r_i$ is the rank function of $M_i$.

- Now, if all matroids are the same $M_i = M$ for all $i$, we get condition

  $$|A| \leq k r(A) \quad \forall A \subseteq E$$  \hspace{1cm} (13.2)

- But considering vector of all ones $\mathbf{1} \in \mathbb{R}^E$, this is the same as

  $$\frac{1}{k} |A| = \frac{1}{k} \mathbf{1}(A) \leq r(A) \quad \forall A \subseteq E$$  \hspace{1cm} (13.3)

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem ??)

**Theorem 13.2.1**

If $f : 2^E \rightarrow \mathbb{R}_+$ is given, and $P$ is a polytope in $\mathbb{R}_{+}^E$ of the form

$P = \{ x \in \mathbb{R}_{+}^E : x(A) \leq f(A), \forall A \subseteq E \}$, then the greedy solution to the problem $\max (wx : x \in P)$ is $\forall w$ optimum iff $f$ is monotone non-decreasing submodular (i.e., iff $P$ is a polymatroid).
\[ Pf = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \} \quad (13.6) \]
\[ B_f = Pf \cap \{ x \in \mathbb{R}^E : x(E) = f(E) \} \quad (13.7) \]

**Polymatroid extreme points**

**Theorem 13.2.1**

For a given ordering \( E = (e_1, \ldots, e_m) \) of \( E \) and a given \( E_i = (e_1, \ldots, e_i) \) and \( x \) generated by \( E_i \) using the greedy procedure \( x(e_i) = f(e_i | E_{i-1}) \), then \( x \) is an extreme point of \( Pf \).

**Proof.**

- We already saw that \( x \in Pf \) (Theorem ??).
- To show that \( x \) is an extreme point of \( Pf \), note that it is the unique solution of the following system of equations

  \[ x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \quad (13.10) \]
  \[ x(e) = 0 \text{ for } e \in E \setminus E_i \quad (13.11) \]

  There are \( i \leq m \) equations and \( i \leq m \) unknowns, and simple Gaussian elimination gives us back the \( x \) constructed via the Greedy algorithm!!
Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

**Corollary 13.2.2**

If $x$ is an extreme point of $P_f$ and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \bigcup (A : x(A) = f(A)) = \text{sat}(x)$, then $x$ is generated using greedy by some ordering of $B$.

- Note, $\text{sat}(x) = \text{cl}(x) = \bigcup (A : x(A) = f(A))$ is also called the closure of $x$ (recall that sets $A$ such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem ??)
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of $x$.
- For arbitrary $x$, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A + e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.
Minimizers of a Submodular Function form a lattice

**Theorem 13.2.2**

For arbitrary submodular $f$, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \text{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of $f$. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

**Proof.**

Since $A$ and $B$ are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$. By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)\quad (13.9)$$

Hence, we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$.

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

---

The **sat function** = Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $D(x)$, also called the polymatroid closure or sat (saturation function).
- For some $x \in P_f$, we have defined:

$$\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \overset{\text{def}}{=} \bigcup \{A : A \in D(x)\} = \bigcup \{A : A \subseteq E, x(A) = f(A)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\}\quad (13.9)$$

- Hence, sat($x$) is the maximal (zero-valued) minimizer of the submodular function $f_x(A) \overset{\Delta}{=} f(A) - x(A)$.
- Eq. (13.11) says that sat consists of any point $x$ that is $P_f$ saturated (any additional positive movement, in that dimension, leaves $P_f$). We’ll revisit this in a few slides.
- First, we see how sat generalizes matroid closure.
The sat function = Polymatroid Closure

- Consider matroid \((E, \mathcal{I}) = (E, r)\), some \(I \in \mathcal{I}\). Then \(1_I \in P_r\) and
  \[
  \mathcal{D}(1_I) = \{A : 1_I(A) = r(A)\}
  \]  
  and
  \[
  \text{sat}(1_I) = \bigcup \{A : A \subseteq E, A \in \mathcal{D}(1_I)\}
  \]
  \[
  = \bigcup \{A : A \subseteq E, 1_I(A) = r(A)\}
  \]
  \[
  = \bigcup \{A : A \subseteq E, |I \cap A| = r(A)\}
  \]  

- Notice that \(1_I(A) = |I \cap A| \leq |I|\).
- Intuitively, consider an \(A \supset I \in \mathcal{I}\) that doesn’t increase rank, meaning \(r(A) = r(I)\). If \(r(A) = |I \cap A| = r(I \cap A)\), as in Eqn. (13.4), then \(A\) is in \(I\)’s span, so should get \(\text{sat}(1_I) = \text{span}(I)\).
- We formalize this next.

---

Lemma 13.3.1 (Matroid sat : \(\mathbb{R}^E_+ \rightarrow 2^E\) is the same as closure.)

For \(I \in \mathcal{I}\), we have \(\text{sat}(1_I) = \text{span}(I)\) (13.5)

**Proof.**

- For \(1_I(I) = |I| = r(I)\), so \(I \in \mathcal{D}(1_I)\) and \(I \subseteq \text{sat}(1_I)\). Also, \(I \subseteq \text{span}(I)\).
- Consider some \(b \in \text{span}(I) \setminus I\).
- Then \(I \cup \{b\} \in \mathcal{D}(1_I)\) since \(1_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I)\).
- Thus, \(b \in \text{sat}(1_I)\).
- Therefore, \(\text{sat}(1_I) \supseteq \text{span}(I)\).

...
The sat function $= \text{Polymatroid Closure}$

...proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in D(I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in I$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$.
- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\text{sat}(1_I) \subseteq \text{span}(I)$.
- Hence $\text{sat}(1_I) = \text{span}(I)$.

Prof. Jeff Bilmes
EE596b/Spring 2014/Submodularity - Lecture 13 - May 14th, 2014
F17/38 (pg.17/38)

Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.
- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
- Note $\text{span}(C) = \text{span}(B)$ where $\mathcal{I} \ni B \in \mathcal{B}(C)$ is a base of $C$.
- Then we have $1_B \leq 1_C \leq 1_{\text{span}(C)}$, and that $1_B \in P_r$. We can then make the definition:

$$\text{sat}(1_C) \triangleq \text{sat}(1_B) \text{ for } B \in \mathcal{B}(C) \quad (13.6)$$

In which case, we also get $\text{sat}(1_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(P\text{-basis}(y))$).
- However, consider the following form

$$\text{sat}(1_C) = \bigcup \{A : A \subseteq E, |A \cap C| = r(A)\} \quad (13.7)$$

Exercise: is $\text{span}(C) = \text{sat}(1_C)$? Prove or disprove it.
The sat function, span, and submodular function minimization

- Thus, for a matroid, $\text{sat}(I)$ is exactly the closure (or span) of $I$ in the matroid. I.e., for matroid $(E, r)$, we have $\text{span}(I) = \text{sat}(1_B)$.
- Recall, for $x \in P_f$ and polymatroidal $f$, $\text{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, $\text{span}(I)$ is the maximal minimizer of the submodular function formed by $r(A) - 1_I(A)$.
- Submodular function minimization can solve “span” queries in a matroid or “sat” queries in a polymatroid.

We are given an $x \in P_f^+$ for submodular function $f$.
- Recall that for such an $x$, $\text{sat}(x)$ is defined as

\[
\text{sat}(x) = \bigcup \{ A : x(A) = f(A) \} \tag{13.8}
\]

- We also have stated that $\text{sat}(x)$ can be defined as:

\[
\text{sat}(x) = \left\{ e : \forall \alpha > 0, x + \alpha e \notin P_f^+ \right\} \tag{13.9}
\]
- We next show more formally that these are the same.
sat, as tight polymatroidal elements

-Lets start with one definition and derive the other.

$$\text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha e \notin P^+ \right\}$$ \hspace{1cm} (13.10)

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha e)(A) > f(A) \right\}$$ \hspace{1cm} (13.11)

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists \exists \ e \text{ s.t. } (x + \alpha e)(A) > f(A) \right\}$$ \hspace{1cm} (13.12)

-This last bit follows since $1_e(A) = 1 \iff e \in A$. Continuing, we get

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\}$$ \hspace{1cm} (13.13)

-This is because a tight set $x \in P^+$, meaning $x(A) \leq f(A)$ for all A, we must have

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \right\}$$ \hspace{1cm} (13.14)

$$\text{sat}(x) = \left\{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \right\}$$ \hspace{1cm} (13.15)

-So now, if $A$ is any set such that $x(A) = f(A)$, then we clearly have

$$\forall e \in A, e \in \text{sat}(x), \text{ and therefore that } \text{sat}(x) \supseteq A$$ \hspace{1cm} (13.16)

...and therefore, with sat as defined in Eq. (??),

$$\text{sat}(x) \supseteq \bigcup \left\{ A : x(A) = f(A) \right\}$$ \hspace{1cm} (13.17)

- On the other hand, for any $e \in \text{sat}(x)$ defined as in Eq. (13.15), since $e$ is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

$$\text{sat}(x) \subseteq \bigcup \left\{ A : x(A) = f(A) \right\}$$ \hspace{1cm} (13.18)

- Therefore, the two definitions of sat are identical.
Saturation Capacity

- Another useful concept is saturation capacity which we develop next.
- For \( x \in P_f \), and \( e \in E \), consider finding
  \[
  \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f \} \tag{13.19}
  \]
- This is identical to:
  \[
  \max \{ \alpha : (x + \alpha \mathbf{1}_e)(A) \leq f(A), \forall A \supseteq \{e\} \} \tag{13.20}
  \]
  since any \( B \subseteq E \) such that \( e \notin B \) does not change in a \( \mathbf{1}_e \)
  adjustment, meaning \( (x + \alpha \mathbf{1}_e)(B) = x(B) \).
- Again, this is identical to:
  \[
  \max \{ \alpha : x(A) + \alpha \leq f(A), \forall A \supseteq \{e\} \} \tag{13.21}
  \]
  or
  \[
  \max \{ \alpha : \alpha \leq f(A) - x(A), \forall A \supseteq \{e\} \} \tag{13.22}
  \]
  The max is achieved when
  \[
  \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \tag{13.23}
  \]
  \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \)
  and \( e \).
- Thus we have for \( x \in P_f \),
  \[
  \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \tag{13.24}
  \]
  \[
  = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f \} \tag{13.25}
  \]
- We immediately see that for \( e \in E \setminus \text{sat}(x) \), we have that
  \( \hat{c}(x; e) > 0 \).
- Also, for \( e \in \text{sat}(x) \), we have that \( \hat{c}(x; e) = 0 \).
- Note that any \( \alpha \) with \( 0 \leq \alpha \leq \hat{c}(x; e) \) we have \( x + \alpha \mathbf{1}_e \in P_f \).
- We also see that computing \( \hat{c}(x; e) \) is a form of submodular function
  minimization.
Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given \( x \in P_f \), and \( e \in \text{sat}(x) \), define

\[
\mathcal{D}(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \\
= \mathcal{D}(x) \cap \{ A : A \subseteq E, e \in A \} 
\]  

(13.26)  

(13.27)

- Thus, \( \mathcal{D}(x, e) \subseteq \mathcal{D}(x) \), and \( \mathcal{D}(x, e) \) is a sublattice of \( \mathcal{D}(x) \).
- Therefore, we can define a unique minimal element of \( \mathcal{D}(x, e) \) denoted as follows:

\[
\text{dep}(x, e) = \begin{cases} 
\bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\
\emptyset & \text{else}
\end{cases}
\]  

(13.28)

- I.e., \( \text{dep}(x, e) \) is the minimal element in \( \mathcal{D}(x) \) that contains \( e \) (the minimal \( x \)-tight set containing \( e \)).

Given some \( x \in P_f \),

- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, \( \bigcap_e \text{dep}(x, e) = \text{dep}(x) \).
dep and sat in a lattice

- Given \( x \in P_f \), recall distributive lattice of tight sets \( \mathcal{D}(x) = \{ A : x(A) = f(A) \} \)
- We had that \( \text{sat}(x) = \bigcup \{ A : A \in \mathcal{D}(x) \} \) is the “1” element of this lattice.
- Consider the “0” element of \( \mathcal{D}(x) \), i.e., \( \text{dry}(x) \stackrel{\text{def}}{=} \bigcap \{ A : A \in \mathcal{D}(x) \} \)
- We can see \( \text{dry}(x) \) as the elements that are necessary for tightness.
- That is, we can equivalently define \( \text{dry}(x) \) as
  \[
  \text{dry}(x) = \{ e' : x(A) < f(A), \forall A \not\ni e' \} \quad (13.29)
  \]
- This can be read as, for any \( e' \in \text{dry}(x) \), any set that does not contain \( e' \) is not tight for \( x \) (any set \( A \) that is missing any element of \( \text{dry}(x) \) is not tight).
- Perhaps, then, a better name for \( \text{dry} \) is \( \text{ntight}(x) \), for the necessary for tightness (but we’ll actually use neither name).
- Note that \( \text{dry} \) need not be the empty set. Exercise: give example.

An alternate expression for \( \text{dep} = \text{dry} \)

- Now, given \( x \in P_f \), and \( e \in \text{sat}(x) \), recall distributive sub-lattice of \( e \)-containing tight sets \( \mathcal{D}(x, e) = \{ A : e \in A, x(A) = f(A) \} \)
- We can define the “1” element of this sub-lattice as
  \[
  \text{sat}(x, e) \stackrel{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x, e) \}.
  \]
- Analogously, we can define the “0” element of this sub-lattice as
  \[
  \text{dry}(x, e) \stackrel{\text{def}}{=} \bigcap \{ A : A \in \mathcal{D}(x, e) \}.
  \]
- We can see \( \text{dry}(x, e) \) as the elements that are necessary for \( e \)-containing tightness, with \( e \in \text{sat}(x) \).
- That is, we can view \( \text{dry}(x, e) \) as
  \[
  \text{dry}(x, e) = \{ e' : x(A) < f(A), \forall A \not\ni e', e \in A \} \quad (13.30)
  \]
- This can be read as, for any \( e' \in \text{dry}(x, e) \), any \( e \)-containing set that does not contain \( e' \) is not tight for \( x \).
- But actually, \( \text{dry}(x, e) = \text{dep}(x, e) \), so we have derived another expression for \( \text{dep}(x, e) \) in Eq. (13.30).
Now, let \((E, I) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

Given \(e \in \text{sat}(1_I) \setminus I\) and then consider an \(A \ni e\) with \(|I \cap A| = r(A)\).

Then \(I \cap A\) serves as a base for \(A\) (i.e., \(I \cap A\) spans \(A\)) and any such \(A\) contains a circuit (i.e., we can add \(e \in A \setminus I\) to \(I \cap A\) w/o increasing rank).

Given \(e \in \text{sat}(1_I) \setminus I\), and consider \(\text{dep}(1_I, e)\), with

\[
\text{dep}(1_I, e) = \bigcap \{A : e \in A \subseteq E, 1_I(A) = r(A)\} = \bigcap \{A : e \in A \subseteq E, |I \cap A| = r(A)\} = \bigcap \{A : e \in A \subseteq E, r(A) - |I \cap A| = 0\}
\]

By SFM lattice, \(\exists\) a unique minimal \(A \ni e\) with \(|I \cap A| = r(A)\).

Thus, \(\text{dep}(1_I, e)\) must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

Therefore, when \(e \in \text{sat}(1_I) \setminus I\), then \(\text{dep}(1_I, e) = C(I, e)\) where \(C(I, e)\) is the unique circuit contained in \(I + e\) in a matroid (the fundamental circuit of \(e\) and \(I\) that we encountered before).

Now, if \(e \in \text{sat}(1_I) \cap I\) with \(I \in \mathcal{I}\), we said that \(C(I, e)\) was undefined (since no circuit is created in this case) and so we defined it as \(C(I, e) = \{e\}\).

In this case, for such an \(e\), we have \(\text{dep}(1_I, e) = \{e\}\) since all such sets \(A \ni e\) with \(|I \cap A| = r(A)\) contain \(e\), but in this case no cycle is created, i.e., \(|I \cap A| \geq |I \cap \{e\}| = r(e) = 1\).

We are thus free to take subsets of \(I\) as \(A\), all of which must contain \(e\), but all of which have rank equal to size.

Also note: in general for \(x \in P_f\) and \(e \in \text{sat}(x)\), we have \(\text{dep}(x, e)\) is tight by definition.
Summary of sat, and dep

- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\}$. That is,
  \[
  \text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in D(x)\} \quad (13.34)
  \]
  \[
  = \bigcup \{A : A \subseteq E, x(A) = f(A)\} \quad (13.35)
  \]
  \[
  = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\} \quad (13.36)
  \]

- For $e \in \text{sat}(x)$, we have $\text{dep}(x,e)$ (fundamental circuit) is the minimal (common) saturated ($x$-tight) set w.r.t. $x$ containing $e$. That is,
  \[
  \text{dep}(x,e) = \begin{cases} 
  \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\
  \emptyset & \text{else} 
  \end{cases} \quad (13.37)
  \]
  \[
  = \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1_e - 1_e') \in P_f\} \quad (13.38)
  \]

Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin I$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha 1_e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
- Recall, we have $C(I,e) \setminus e' \in I$ for $e' \in C(I,e)$. I.e., $C(I,e)$ consists of elements that when removed recover independence.
- In other words, for $e \in \text{span}(I) \setminus I$, we have that
  \[
  C(I,e) = \{a \in E : I + e - a \in I\} \quad (13.38)
  \]
  I.e., an addition of $e$ to $I$ stays within $I$ only if we simultaneously remove one of the elements of $C(I,e)$.
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x,e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x,e)$ property to take the form: a positive move in the $e$-direction stays within $P_f^+$ only if we simultaneously take a negative move in one of the $\text{dep}(x,e)$ directions.
**Dependence Function and exchange in 2D**

- \( \text{dep}(x, e) \) is set of neg. directions we must move if we want to move in pos. \( e \) direction, starting at \( x \) and staying within \( P_f \).
- Viewable in 2D, we have for \( A, B \subseteq E, A \cap B = \emptyset \):

\[
\begin{align*}
\text{Left: } & A \cap \text{dep}(x, e) = \emptyset, \text{ and we can't move further in (e) direction, and moving in any negative } a \in A \text{ direction doesn't change that. Notice no dependence between (e) and any element in A.} \\
\text{Right: } & A \subseteq \text{dep}(x, e), \text{ and we can't move further in the (e) direction, but we can move further in (e) direction by moving in some } a \in A \text{ negative direction. Notice dependence between (e) and elements in A.}
\end{align*}
\]

**Dependence Function and exchange in 3D**

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have:

\[
\begin{align*}
\text{l.e., for } & e \in \text{sat}(x), a \in \text{sat}(x), a \in \text{dep}(x, e), e \notin \text{dep}(x, a), \text{ and } \\
\text{dep}(x, e) &= \{a : a \in E, \exists \alpha > 0 : x + \alpha(1_e - 1_a) \in P_f\} \\
\end{align*}
\] (13.39)

- We next show this formally...
The derivation for $\text{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable $\alpha$:

\[
\text{dep}(x, e) = \text{ntight}(x, e) = \{e' : x(A) < f(A), \forall A \not\ni e', e \in A\} = \{e' : \exists \alpha > 0, \text{ s.t. } \alpha \leq f(A) - x(A), \forall A \not\ni e', e \in A\} = \{e' : \exists \alpha > 0, \text{ s.t. } \alpha(1_e(A) - 1_{e'}(A)) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A) \text{ and } \forall A \not\ni e', e \in A\}
\]

Now, $1_e(A) - 1_{e'}(A) = 0$ if either \{e, e'\} $\subseteq A$, or \{e, e'\} $\cap A = \emptyset$.

Also, if $e' \in A$ but $e \not\in A$, then
\[
x(A) + \alpha(1_e(A) - 1_{e'}(A)) = x(A) - \alpha \leq f(A) \text{ since } x \in P_f.
\]

thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get
\[
\text{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A\}
\]

This is then identical to
\[
\text{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(1_e - 1_{e'}) \in P_f\}
\]

Compare with original, the minimal element of $\mathcal{D}(x, e)$, with $e \in \text{sat}(x)$:
\[
\text{dep}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \text{ if } e \in \text{sat}(x)\]

\[
\emptyset \text{ else}
\]
Summary of Concepts

- Most violated inequality: $\max \{ x(A) - f(A) : A \subseteq E \}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$.
- Minimizers of submodular functions form a lattice.
- Minimal and maximal element of a lattice.
- $x$-tight sets, maximal and minimal tight set.
- $sat$ function & Closure
- Saturation Capacity
- $e$-containing tight sets
- dep function & fundamental circuit of a matroid

Summary important definitions so far: tight, dep, & sat

- $x$-tight sets: For $x \in P_f$, $D(x) = \{ A \subseteq E : x(A) = f(A) \}$.
- Polymatroid closure/maximal $x$-tight set: For $x \in P_f$,
  $sat(x) = \cup \{ A : A \in D(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$.
- Saturation capacity: for $x \in P_f$, $0 \leq \hat{c}(x; e) = \min \{ f(A) - x(A) | \forall A \ni e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}$.
- Recall: $sat(x) = \{ e : \hat{c}(x; e) = 0 \}$ and $E \setminus sat(x) = \{ e : \hat{c}(x; e) > 0 \}$.
- $e$-containing tight sets: For $x \in P_f$,
  $D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \subseteq D(x)$.
- Minimal $e$-containing $x$-tight set/polymatroidal fundamental circuit: For $x \in P_f$,
  $dep(x, e) = \begin{cases} \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in sat(x) \\ \emptyset & \text{else} \end{cases}$
  
  $= \{ e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1_e - 1_{e'}) \in P_f \}$