Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 17 —

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\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B)
\]

Read Tom McCormick’s overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf

Read chapters 1 - 4 from Fujishige book.


Read lecture 14 slides on lattice theory at our web page (http://j. ee.washington.edu/~bilmes/classes/ee596b_spring_2014/)
Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).
L1 (3/31): Motivation, Applications, & Basic Definitions
L2: (4/2): Applications, Basic Definitions, Properties
L3: More examples and properties (e.g., closure properties), and examples, spanning trees
L4: proofs of equivalent definitions, independence, start matroids
L5: matroids, basic definitions and examples
L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
L7: Dual Matroids, other matroid properties, Combinatorial Geometries
L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
L9: From Matroid Polytopes to Polymatroids.
L10: Polymatroids and Submodularity
L11: More properties of polymatroids, SFM special cases
L12: polymatroid properties, extreme points polymatroids,
L13: sat, dep, supp, exchange capacity, examples
L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
L16: minimum norm point algorithm and the lattice of minimizers of a submodular function, Lovasz extension
L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
L18:
L19:
L20:

Finals Week: June 9th-13th, 2014.
**Min-Norm Point and SFM**

**Theorem 17.2.1**

Let $y^*$, $A_-$, and $A_0$ be as given. Then $y^*$ is a maximizer of the l.h.s. of Eqn. (??). Moreover, $A_-$ is the unique minimal minimizer of $f$ and $A_0$ is the unique maximal minimizer of $f$.

**Proof.**

- First note, since $x^* \in B_f$, we have $x^*(E) = f(E)$, meaning $\text{sat}(x^*) = E$. Thus, we can consider any $e \in E$ within $\text{dep}(x^*, e)$.

- Consider any pair $(e, e')$ with $e' \in \text{dep}(x^*, e)$ and $e \in A_-$. Then $x^*(e) < 0$, and $\exists \alpha > 0$ s.t. $x^* + \alpha 1_e - \alpha 1_{e'} \in P_f$.

- We have $x^*(E) = f(E)$ and $x^*$ is minimum in $l_2$ sense. We have $(x^* + \alpha 1_e - \alpha 1_{e'}) \in P_f$, and in fact

\[
(x^* + \alpha 1_e - \alpha 1_{e'})(E) = x^*(E) + \alpha - \alpha = f(E)
\]

so $x^* + \alpha 1_e - \alpha 1_{e'} \in B_f$ also.

...
Recall, that the set of minimizers of $f$ forms a lattice.

In fact, with $x^*$ the min-norm point, and $A_-$ and $A_0$ as defined above, we have the following theorem:

**Theorem 17.2.1**

Let $A \subseteq E$ be any minimizer of submodular $f$, and let $x^*$ be the minimum-norm point. Then $A$ has the form:

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$$

(17.7)

for some set $A_m \subseteq A_0 \setminus A_-$. 

for some set $A_m \subseteq A_0 \setminus A_-$. 
A continuous extension of submodular $f$

That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, and defining $E_i = \{e_1, e_2, \ldots, e_i\}$ and where we choose the element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$, we have

$$\tilde{f}(w) = \max(wx : x \in P_f)$$

(17.11)

$$= \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1})$$

(17.12)

$$= \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1}))$$

(17.13)

$$= w(e_m) f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1})) f(E_i)$$

(17.14)

We say that $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ forms a chain based on $w$. 

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A continuous extension of submodular $f$

- Definition of the continuous extension, once again, for reference:

$$\tilde{f}(w) = \max(wx : x \in P_f)$$ (17.11)

- Therefore, if $f$ is a submodular function, we can write

$$\tilde{f}(w) = w(e_m)f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(E_i)$$ (17.12)

$$= \sum_{i=1}^{m} \lambda_i f(E_i)$$ (17.13)

where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted according to $w$ as before.

- From convex analysis, we know $\tilde{f}(w) = \max(wx : x \in P)$ is always convex in $w$ for any set $P \subseteq \mathbb{R}^E$, since it is the maximum of a set of linear functions (true even when $f$ is not submodular or $P$ is not a convex set).
An extension of an arbitrary $f : 2^V \to \mathbb{R}$

Thus, for any $f : 2^E \to \mathbb{R}$, even non-submodular $f$, we can define an extension, having $\tilde{f}(1_A) = f(A)$, $\forall A$, in this way where

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i)$$

(17.20)

with the $E_i = \{e_1, \ldots, e_i\}$’s defined based on sorted descending order of $w$ as in $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$, and where

for $i \in \{1, \ldots, m\}$, $\lambda_i = \begin{cases} w(e_i) - w(e_{i+1}) & \text{if } i < m \\ w(e_m) & \text{if } i = m \end{cases}$

(17.21)

so that $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$.

$w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$ is an interpolation of certain hypercube vertices.

$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i)$ is the associated interpolation of the values of $f$ at sets corresponding to each hypercube vertex.
Summary: comparison of the two extension forms

- So if $f$ is submodular, then we can write $\tilde{f}(w) = \max(wx : x \in P_f)$ (which is clearly convex) in the form:

$$\tilde{f}(w) = \max(wx : x \in P_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \quad (17.1)$$

where $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$ and $E_i = \{e_1, \ldots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. 

On the other hand, for any $f$ (even non-submodular), we can produce an extension $\tilde{f}$ having the form

$$\tilde{f}(w) = \max(wx : x \in P_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \quad (17.2)$$

where $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$ and $E_i = \{e_1, \ldots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. 

In both Eq. (17.1) and Eq. (17.2), we have $\tilde{f}(1_A) = f(A)$, $\forall A$, but Eq. (17.2) might not be convex. Submodularity is sufficient for convexity but is it necessary?
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- In both Eq. (17.1) and Eq. (17.2), we have \( \tilde{f}(1_A) = f(A), \ \forall A, \) but Eq. (17.2), might not be convex.

- Submodularity is sufficient for convexity of but is it necessary?
Theorem 17.2.1

A function \( f : 2^E \to \mathbb{R} \) is submodular iff its Lovász extension \( \tilde{f} \) of \( f \) is convex.

Proof.

- We’ve already seen that if \( f \) is submodular, its extension can be written via Eqn. (??) due to the greedy algorithm, and therefore is also equivalent to \( \tilde{f}(w) = \max \{wx : x \in P_f\} \), and thus is convex.

- Conversely, suppose the Lovász extension \( \tilde{f}(w) = \sum \lambda_i f(E_i) \) of some function \( f : 2^E \to \mathbb{R} \) is a convex function.

- We note that, based on the extension definition, in particular the definition of the \( \{\lambda_i\}_i \), we have that \( \tilde{f}(\alpha w) = \alpha \tilde{f}(w) \) for any \( \alpha \in \mathbb{R}_+ \). I.e., \( f \) is a positively homogeneous convex function.

...
Integration and Aggregation

- Integration is just summation (e.g., the $\int$ symbol has as its origins a sum).
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- Lebesgue integration allows integration w.r.t. an underlying measure $\mu$ of sets. E.g., given measurable function $f$, we can define

$$\int_X f \, d\mu = \sup I_X(s)$$

where $I_X(s) = \sum_{i=1}^{n} c_i \mu(X \cap X_i)$, and where we take the sup over all measurable functions $s$ such that $0 \leq s \leq f$ and $s(x) = \sum_{i=1}^{n} c_i I_{X_i}(x)$ and where $I_{X_i}(x)$ is indicator of membership of set $X_i$, with $c_i > 0$. 
In finite discrete spaces, Lebesgue integration is just a weighted average, and can be seen as an aggregation function.
Integration, Aggregation, and Weighted Averages

- In finite discrete spaces, Lebesgue integration is just a weighted average, and can be seen as an aggregation function.
- I.e., given a weight vector $w \in [0, 1]^E$ for some finite ground set $E$, then for any $x \in \mathbb{R}^E$ we have

$$\text{WAVG}(x) = \sum_{e \in E} x(e)w(e)$$  \hspace{1cm} (17.4)
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Consider \( 1_e \) for \( e \in E \), we have

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\text{WAVG}(1_e) = w(e) \tag{17.5}
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  so seen as a function on the hypercube vertices, the entire \( \text{WAVG} \) function is given based on values on a size \( m = |E| \) subset of the vertices of this hypercube, i.e., \( \{1_e : e \in E\} \).
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  so seen as a function on the hypercube vertices, the entire WAVG function is given based on values on a size $m = |E|$ subset of the vertices of this hypercube, i.e., \{1_e : e \in E\}. Moreover, we are interpolating as in
  \[
  \text{WAVG}(x) = \sum_{e \in E} x(e)w(e) = \sum_{e \in E} x(e)\text{WAVG}(1_e) \tag{17.6}
  \]
Integration, Aggregation, and Weighted Averages

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  \]
- Note, WAVG function is linear in the weights \( w \), and homogeneous.
  \[
  \text{WAVG}_{w_1 + w_2}(x) = \text{WAVG}_{w_1}(x) + \text{WAVG}_{w_2}(x), \text{WAVG}(\alpha x) = \alpha \text{WAVG}(x). \]
More complex “nonlinear” aggregation functions can be constructed by defining the aggregation function on all vertices of the hypercube. I.e., for each $1_A : A \subseteq E$ we might have (for all $A \subseteq E$):

$$AG(1_A) = w_A$$  \hspace{1cm} (17.7)
Integration, Aggregation, and Weighted Averages

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- What then might $AG(x)$ be for some $x \in \mathbb{R}^E$? Our weighted average functions might look something more like the r.h.s. in:

$$AG(x) = \sum_{A \subseteq E} x(A)w_A = \sum_{A \subseteq E} x(A)AG(1_A)$$  \hfill (17.8)
Integration, Aggregation, and Weighted Averages

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- Note, we can define $w(e) = w'(e)$ and $w(A) = 0, \forall A : |A| > 1$ and get back previous (normal) weighted average, in that

$$\text{WAVG}_{w'}(x) = AG_w(x)$$  \hspace{1cm} (17.9)
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Set function $f : 2^E \rightarrow \mathbb{R}$ is a game if $f$ is normalized $f(\emptyset) = 0$. 

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Set function $f : 2^E \rightarrow \mathbb{R}$ is called a capacity if it is monotone non-decreasing, i.e., $f(A) \leq f(B)$ whenever $A \subseteq B$. 
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A Boolean function $f$ is any function $f : \{0, 1\}^m \rightarrow \{0, 1\}$ and is a pseudo-Boolean function if $f : \{0, 1\}^m \rightarrow \mathbb{R}$.
Set function $f : 2^E \rightarrow \mathbb{R}$ is called a capacity if it is monotone non-decreasing, i.e., $f(A) \leq f(B)$ whenever $A \subseteq B$.

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Any set function corresponds to a pseudo-Boolean function. I.e., given $f : 2^E \rightarrow \mathbb{R}$, form $f_b : \{0, 1\}^m \rightarrow \mathbb{R}$ as $f_b(x) = f(A_x)$ where the $A, x$ bijection is $A = \{e \in E : x_e = 1\}$ and $x = 1_A$. 
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Also, if we have an expression for $f_b$ we can construct a set function $f$ as $f(A) = f_b(1_A)$. We can also often relax $f_b$ to any $x \in [0, 1]^m$. 
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We saw this for Lovász extension.
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Any set function corresponds to a pseudo-Boolean function. I.e., given \( f : 2^E \to \mathbb{R} \), form \( f_b : \{0, 1\}^m \to \mathbb{R} \) as \( f_b(x) = f(A_x) \) where the \( A, x \) bijection is \( A = \{ e \in E : x_e = 1 \} \) and \( x = 1_A \).

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It turns out that a concept essentially identical to the Lovász extension was derived much earlier, in 1954, and using this derivation (via integration) leads to deeper intuition.
Definition 17.3.1

Let \( f \) be any capacity on \( E \) and \( w \in \mathbb{R}^E_+ \). The Choquet integral (1954) of \( w \) w.r.t. \( f \) is defined by

\[
C_f(w) = \sum_{i=1}^{m} (w_{e_i} - w_{e_{i+1}}) f(E_i)
\]  

(17.10)

where in the sum, we have sorted and renamed the elements of \( E \) so that \( w_{e_1} \geq w_{e_2} \geq \cdots \geq w_{e_m} \geq w_{e_{m+1}} = 0 \), and where \( E_i = \{e_1, e_2, \ldots, e_i\} \).

- We immediately see that an equivalent formula is as follows:

\[
C_f(w) = \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1}))
\]  

(17.11)

where \( E_0 \overset{\text{def}}{=} \emptyset \).
**Definition 17.3.1**

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$$C_f(w) = \sum_{i=1}^{m} (w_{e_i} - w_{e_{i+1}}) f(E_i)$$

(17.10)

where in the sum, we have sorted and renamed the elements of $E$ so that $w_{e_1} \geq w_{e_2} \geq \cdots \geq w_{e_m} \geq w_{e_{m+1}} = 0$, and where $E_i = \{e_1, e_2, \ldots, e_i\}$.

- BTW: this again essentially **Abel’s partial summation formula**: Given two arbitrary sequences $\{a_n\}$ and $\{b_n\}$ with $A_n = \sum_{k=1}^{n} a_k$, we have

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1} - A_{m-1} b_m$$

(17.12)
The “integral” in the Choquet integral

- Thought of as an integral over $\mathbb{R}$ of a piece-wise constant function.
The “integral” in the Choquet integral

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- First note, assuming $E$ is ordered according to descending $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_{m-1}) \geq w(e_m)$, then $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e \geq w_{e_i}\}$. 

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  $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e \geq w_{e_i}\}$.

- For any $w_{e_i} > \alpha \geq w_{e_{i+1}}$ we also have
  
  $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e > \alpha\}$. 
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- For any $w_{e_i} > \alpha \geq w_{e_{i+1}}$ we also have $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e > \alpha\}$.
- Consider segmenting the real-axis at boundary points $w_{e_i}$, right most is $w_{e_1}$.

\[ w(e_m) \quad w(e_{m-1}) \quad \cdots \quad w(e_5) \quad w(e_4) \quad w(e_3) \quad w(e_2) \quad w(e_1) \]
The “integral” in the Choquet integral

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- Consider segmenting the real-axis at boundary points $w_{e_i}$, right most is $w_{e_1}$.

\[
\begin{align*}
  &w(e_m) \quad w(e_{m-1}) \quad \cdots \quad w(e_5) \quad w(e_4) \quad w(e_3) \quad w(e_2) \quad w(e_1) \\
\end{align*}
\]

- A function can be defined on a segment of $\mathbb{R}$, namely
  \[ w_{e_i} > \alpha \geq w_{e_{i+1}}. \] This function $F_i : [w_{e_{i+1}}, w_{e_i}) \rightarrow \mathbb{R}$ is defined as
  \[ F_i(\alpha) = f(\{e \in E : w_e > \alpha\}) = f(E_i) \quad (17.13) \]
The “integral” in the Choquet integral

We can generalize this to multiple segments of \( \mathbb{R} \) (for now, take \( w \in \mathbb{R}^E_+ \)). The piecewise-constant function is defined as:

\[
F(\alpha) = \begin{cases} 
    f(E) & \text{if } 0 \leq \alpha < w_m \\
    f(\{e \in E : w_e > \alpha\}) & \text{if } w_{e_{i+1}} \leq \alpha < w_{e_i}, \ i \in \{1, \ldots, m-1\} \\
    0 & \text{if } w_1 < \alpha 
\end{cases}
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- We can generalize this to multiple segments of $\mathbb{R}$ (for now, take $w \in \mathbb{R}_+^E$). The piecewise-constant function is defined as:

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    0 & \text{if } w_1 < \alpha
\end{cases}$$

- Visualizing a piecewise constant function, where the constant values are given by $f$ evaluated on $E_i$ for each $i$

Note, what is depicted may be a game but not a capacity. Why?
Now consider the integral, with $w \in \mathbb{R}^E_+$, and normalized $f$ so that $f(\emptyset) = 0$. Recall $w_{m+1} \overset{\text{def}}{=} 0$.

\[ \tilde{f}(w) \overset{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \]  

(17.14)
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$$\tilde{f}(w) \overset{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha$$  \hspace{1cm} (17.14)

$$= \int_0^\infty f(\{e \in E : w_e > \alpha\}) d\alpha$$ \hspace{1cm} (17.15)
The “integral” in the Choquet integral

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\[
\tilde{f}(w) \overset{\text{def}}{=} \int_{0}^{\infty} F(\alpha)d\alpha = \int_{0}^{\infty} f(\{e \in E : w_e > \alpha\})d\alpha = \int_{w_{m+1}}^{\infty} f(\{e \in E : w_e > \alpha\})d\alpha = \sum_{i=1}^{m} \int_{w_i}^{w_{i+1}} f(\{e \in E : w_e > \alpha\})d\alpha
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Now consider the integral, with $w \in \mathbb{R}^E_+$, and normalized $f$ so that $f(\emptyset) = 0$. Recall $w_{m+1} \overset{\text{def}}{=} 0$.

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\tilde{f}(w) \overset{\text{def}}{=} \int_0^{\infty} F(\alpha) d\alpha \\
= \int_0^{\infty} f(\{e \in E : w_e > \alpha\}) d\alpha \quad (17.15) \\
= \int_{w_{m+1}}^{\infty} f(\{e \in E : w_e > \alpha\}) d\alpha \quad (17.16) \\
= \sum_{i=1}^{m} \int_{w_{i+1}}^{w_i} f(\{e \in E : w_e > \alpha\}) d\alpha \quad (17.17) \\
= \sum_{i=1}^{m} \int_{w_{i+1}}^{w_i} f(E_i) d\alpha = \sum_{i=1}^{m} f(E_i)(w_i - w_{i+1}) \quad (17.18)
$$
The “integral” in the Choquet integral

- But we saw before that $\sum_{i=1}^{m} f(E_i)(w_i - w_{i+1})$ is just the Lovász extension of a function $f$. 

Thus, we have the following definition:

**Definition 17.3.2**

Given $w \in \mathbb{R}_{E}^+$, the Lovász extension (equivalently Choquet integral) may be defined as follows:

$$\tilde{f}(w) \overset{\text{def}}{=} \int_{0}^{\infty} F(\alpha) d\alpha \quad (17.19)$$

where the function $F$ is defined as before.

Note that it is not necessary in general to require $w \in \mathbb{R}_{E}^+$ (i.e., we can take $w \in \mathbb{R}_{E}$) nor that $f$ be non-negative, but it is a bit more involved. Above is the simple case.
The “integral” in the Choquet integral

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- Note that it is not necessary in general to require \( w \in \mathbb{R}_+^E \) (i.e., we can take \( w \in \mathbb{R}^E \)) nor that \( f \) be non-negative, but it is a bit more involved. Above is the simple case.
Recall, we want to produce some notion of generalized aggregation function having the flavor of:

$$\text{AG}(x) = \sum_{A \subseteq E} x(A)w_A = \sum_{A \subseteq E} x(A)\text{AG}(1_A)$$  \hspace{1cm} (17.20)

how does this correspond to Lovász extension?
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E.g., for each $i$, $\mathcal{V}_i = \{1_{A_1}, 1_{A_2}, \ldots, 1_{A_k}\}$ ($k$ vertices) and the convex hull of $V_i$ defines the $i^{th}$ polytope.
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This forms a “triangulation” of the hypercube.
Choquet integral and aggregation

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- This forms a “triangulation” of the hypercube.

- For any \(x \in [0, 1]^m\) there is a (not necessarily unique) \(\mathcal{V}(x) = V_j\) for some \(j\) such that \(x \in \text{conv}(\mathcal{V}(x))\).
Most generally, for $x \in [0,1]^m$, let us define the (unique) coefficients $\alpha_0^x(A)$ and $\alpha_i^x(A)$ that define the affine transformation of the coefficients of $x$ to be used with the particular hypercube vertex $1_A$. The affine transformation is as follows:

\[
\alpha_0^x(A) + \sum_{j=1}^{m} \alpha_j^x(A)x_j \in \mathbb{R} \tag{17.21}
\]

Note that many of these coefficient are often zero.
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(17.21)

Note that many of these coefficient are often zero.

From this, we can define an aggregation function of the form

$$
AG(x) \overset{\text{def}}{=} \sum_{A:1_A \in \mathcal{V}(x)} \left( \alpha_0^x(A) + \sum_{j=1}^{m} \alpha_j^x(A)x_j \right)AG(1_A) 
$$

(17.22)
We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation $\sigma$, define

$$\text{conv}(\mathcal{V}_\sigma) = \left\{ x \in [0, 1]^n \mid x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(m)} \right\}$$

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Then these $m!$ blocks of the partition are called the canonical partitions of the hypercube.
Choquet integral and aggregation

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- With this, we can define $\{\mathcal{V}_i\}_i$ as the vertices of $\text{conv}(\mathcal{V}_\sigma)$ for each permutation $\sigma$. 

Hence, Lovász extension is a generalized aggregation function.
Choquet integral and aggregation

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- With this, we can define $\{\mathcal{V}_i\}_i$ as the vertices of $\text{conv}(\mathcal{V}_\sigma)$ for each permutation $\sigma$. In this case, we have:
Choquet integral and aggregation

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**Proposition 17.3.3**

The above linear interpolation in Eqn. (17.22) using the canonical partition yields the Lovász extension with

$$\alpha^x_0(A) + \sum_{j=1}^m \alpha^x_j(A)x_j = x_{\sigma_i} - x_{\sigma_{i-1}} \text{ for } A = E_i = \{e_{\sigma_1}, \ldots, e_{\sigma_i}\} \text{ for appropriate order } \sigma.$$
We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation $\sigma$, define

$$\text{conv}(\mathcal{V}_\sigma) = \left\{ x \in [0, 1]^n \big| x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(m)} \right\} \quad (17.23)$$

Then these $m!$ blocks of the partition are called the canonical partitions of the hypercube.

With this, we can define $\{\mathcal{V}_i\}_i$ as the vertices of $\text{conv}(\mathcal{V}_\sigma)$ for each permutation $\sigma$. In this case, we have:

**Proposition 17.3.3**

*The above linear interpolation in Eqn. (17.22) using the canonical partition yields the Lovász extension with $\alpha_x^0(A) + \sum_{j=1}^{m} \alpha_x^j(A)x_j = x_{\sigma_i} - x_{\sigma_{i-1}}$ for $A = E_i = \{e_{\sigma_1}, \ldots, e_{\sigma_i}\}$ for appropriate order $\sigma$.***

Hence, Lovász extension is a generalized aggregation function.
Lovász extension as max over orders

We can also write the Lovász extension as follows:

\[ \tilde{f}(w) = \max_{\sigma \in \Pi[m]} w^\top c^\sigma \]  

(17.24)

where \( \Pi[m] \) is the set of \( m! \) permutations of \( [m] = E \), \( \sigma \in \Pi[m] \) is a particular permutation, and \( c^\sigma \) is a vector associated with permutation \( \sigma \) defined as:

\[ c^\sigma_i = f(E_{\sigma_i}) - f(E_{\sigma_{i-1}}) \]  

(17.25)

where \( E_{\sigma_i} = \{e_{\sigma_1}, e_{\sigma_2}, \ldots, e_{\sigma_i}\} \).
Lovász extension as max over orders

- We can also write the Lovász extension as follows:

\[ \tilde{f}(w) = \max_{\sigma \in \Pi[m]} w^T c^\sigma \quad (17.24) \]

where \( \Pi[m] \) is the set of \( m! \) permutations of \( [m] = E \), \( \sigma \in \Pi[m] \) is a particular permutation, and \( c^\sigma \) is a vector associated with permutation \( \sigma \) defined as:

\[ c^\sigma_i = f(E_{\sigma_i}) - f(E_{\sigma_i-1}) \quad (17.25) \]

where \( E_{\sigma_i} = \{ e_{\sigma_1}, e_{\sigma_2}, \ldots, e_{\sigma_i} \} \).

- Note this immediately follows from the definition of the Lovász extension in the form:

\[ \tilde{f}(w) = \max_{x \in P_f} w^T x = \max_{x \in B_f} w^T x \quad (17.26) \]

since we know that the maximum is achieved by an extreme point of the base \( B_f \) and all extreme points are obtained by a permutation-of-\( E \)-parameterized greedy instance.
Lovász extension, defined in multiple ways

- As shorthand notation, let's use \( \{ w \geq \alpha \} \equiv \{ e \in E : w(e) \geq \alpha \} \), called the weak \( \alpha \)-sup-level set of \( w \).
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- Given any \( w \in \mathbb{R}^E \), sort \( E \) as \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \).
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- Given any \( w \in \mathbb{R}^E \), sort \( E \) as \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \). Also, w.l.o.g., number elements of \( w \) so that \( w_1 \geq w_2 \geq \cdots \geq w_m \).
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- Given any \( w \in \mathbb{R}^E \), sort \( E \) as \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \). Also, w.l.o.g., number elements of \( w \) so that \( w_1 \geq w_2 \geq \cdots \geq w_m \).
- We have already seen how we can define the Lovász extension for any (not necessarily submodular) function \( f \) in the following equivalent ways:

\[
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i)f(e_i | E_{i-1})
\]

(17.27)

\[
= \sum_{i=1}^{m-1} f(E_i)(w(e_i) - w(e_{i+1})) + f(E)w(e_m)a
\]

(17.28)

\[
= \sum_{i=1}^{m-1} \lambda_i f(E_i)
\]

(17.29)
Lovász extension, as integral

- Additional ways we can define the Lovász extension for any (not necessarily submodular) but normalized function \( f \) include:

\[
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} \lambda_i f(E_i) 
\]  \hspace{1cm} (17.30)

\[
= \sum_{i=1}^{m-1} f(E_i)(w(e_i) - w(e_{i+1})) + f(E)w(e_m) 
\]  \hspace{1cm} (17.31)

\[
= \int_{\min\{w_1,...,w_m\}}^{+\infty} f(\{w \geq \alpha\})d\alpha + f(E) \min\{w_1,...,w_m\} 
\]  \hspace{1cm} (17.32)

\[
= \int_{0}^{+\infty} f(\{w \geq \alpha\})d\alpha + \int_{-\infty}^{0} [f(\{w \geq \alpha\}) - f(E)]d\alpha 
\]  \hspace{1cm} (17.33)
In fact, we have that, given function $f$, and any $w \in \mathbb{R}^E$:

$$
\tilde{f}(w) = \int_{-\infty}^{+\infty} \hat{f}(\alpha) d\alpha
$$

(17.34)

where

$$
\hat{f}(\alpha) = \begin{cases} 
  f(\{w \geq \alpha\}) & \text{if } \alpha \geq 0 \\
  f(\{w \geq \alpha\}) - f(E) & \text{if } \alpha < 0 
\end{cases}
$$

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In fact, we have that, given function $f$, and any $w \in \mathbb{R}^E$:

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So we can write it as a simple integral over the right function.
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$$

(17.35)

So we can write it as a simple integral over the right function.

These make it easier to see certain properties of the Lovász extension. But first, we show the above.
To show Eqn. (17.32), first note that the r.h.s. terms are the same since $w(e_m) = \min \{w_1, \ldots, w_m\}$. 
Lovász extension, as integral

- To show Eqn. (17.32), first note that the r.h.s. terms are the same since $w(e_m) = \min \{w_1, \ldots, w_m\}$.

- Then, consider that, as a function of $\alpha$, we have

$$f(\{w \geq \alpha\}) = \begin{cases} 
0 & \text{if } \alpha > w(e_1) \\
f(E_k) & \text{if } \alpha \in (w(e_{k+1}), w(e_k)), k \in \{1, \ldots, m-1\} \\
f(E) & \text{if } \alpha < w(e_m)
\end{cases}$$

(17.36)

we use open intervals since sets of zero measure don’t change integration.
Lovász extension, as integral

- To show Eqn. (17.32), first note that the r.h.s. terms are the same since $w(e_m) = \min \{w_1, \ldots, w_m\}$.

- Then, consider that, as a function of $\alpha$, we have

$$f(\{w \geq \alpha\}) = \begin{cases} 0 & \text{if } \alpha > w(e_1) \\ f(E_k) & \text{if } \alpha \in (w(e_{k+1}), w(e_k)), k \in \{1, \ldots, m - 1\} \\ f(E) & \text{if } \alpha < w(e_m) \end{cases}$$

(17.36)

we use open intervals since sets of zero measure don’t change integration.

- Inside the integral, then, this recovers Eqn. (17.31).
Lovász extension, as integral

To show Eqn. (17.33), start with Eqn. (17.32), note

\[ w_m = \min \{ w_1, \ldots, w_m \}, \text{ take any } \beta \leq \min \{ 0, w_1, \ldots, w_m \}, \text{ and form:} \]

\[ \tilde{f}(w) \]
To show Eqn. (17.33), start with Eqn. (17.32), note

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\tilde{f}(w) = \int_{w_m}^{+\infty} f(\{w \geq \alpha\}) d\alpha + f(E) \min \{ w_1, \ldots, w_m \}
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$$\tilde{f}(w) = \int_{w_m}^{+\infty} f(\{w \geq \alpha\})d\alpha + f(E) \min \{w_1, \ldots, w_m\}$$

$$= \int_{\beta}^{+\infty} f(\{w \geq \alpha\})d\alpha - \int_{\beta}^{w_m} f(\{w \geq \alpha\})d\alpha + f(E) \int_{0}^{w_m} d\alpha$$
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\[
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Lovász extension, as integral

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\[ = \int_{0}^{+\infty} f(\{ w \geq \alpha \}) d\alpha + \int_{-\infty}^{0} [f(\{ w \geq \alpha \}) - f(E)] d\alpha \]
Lovász extension properties

- Using the above, have the following (some of which we’ve seen):

1. **Superposition of LE operator**: Given $f$ and $g$ with Lovász extensions $\tilde{f}$ and $\tilde{g}$ then $\tilde{f} + \tilde{g}$ is the Lovász extension of $f + g$ and $\lambda \tilde{f}$ is the Lovász extension of $\lambda f$ for $\lambda \in \mathbb{R}$.

2. If $w \in \mathbb{R}^E$ then $\tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\}) d\alpha$.

3. For $w \in \mathbb{R}^E$, and $\alpha \in \mathbb{R}$, we have $\tilde{f}(w + \alpha 1_E) = \tilde{f}(w) + \alpha f(E)$.

4. **Positive homogeneity**: I.e., $\tilde{f}(\alpha w) = \alpha \tilde{f}(w)$ for $\alpha \geq 0$.

5. For all $A \subseteq E$, $\tilde{f}(1_A) = f(A)$.

6. $f$ is symmetric as in $f(A) = f(E \setminus A)$, $\forall A$, then $\tilde{f}(w) = \tilde{f}(-w)$ ($\tilde{f}$ is even).

7. Given partition $E_1 \cup E_2 \cup \cdots \cup E_k$ of $E$ and $w = \sum_{i=1}^k \gamma_i 1_{E_k}$ with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$, and with $E_{1:i} = E_1 \cup E_2 \cup \cdots \cup E_i$, then $\tilde{f}(w) = \sum_{i=1}^k \gamma_i f(E_{1:i}) = \sum_{i=1}^{k-1} \gamma_i f(E_{1:i}) (\gamma_i - \gamma_{i+1} + 1) + f(E)$.
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Using the above, have the following (some of which we’ve seen):

**Theorem 17.4.1**

Let \( f, g : 2^E \rightarrow \mathbb{R} \) be normalized (\( f(\emptyset) = g(\emptyset) = 0 \)). Then
Lovász extension properties

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Prof. Jeff Bilmes
Lovász extension properties

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Consider property property 3, for example, which says that
\[ \tilde{f}(w + \alpha 1_E) = \tilde{f}(w) + \alpha f(E). \]
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This means that, say when \( m = 2 \), that as we move along the line \( w_1 = w_2 \), the Lovász extension scales linearly.
Consider property property 3, for example, which says that
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This means that, say when \( m = 2 \), that as we move along the line \( w_1 = w_2 \), the Lovász extension scales linearly.

And if \( f(E) = 0 \), then the Lovász extension is constant along the direction \( 1_E \).
Lovász extension properties

- Given Eqns. (17.30) through (17.33), most of the above properties are relatively easy to derive.
Lovász extension properties

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- For example, if \( f \) is symmetric, and since \( f(E) = f(\emptyset) = 0 \), we have

\[
\tilde{f}(-\omega)
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Lovász extension properties

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- For example, if $f$ is symmetric, and since $f(E) = f(\emptyset) = 0$, we have

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\tilde{f}(-w) = \int_{-\infty}^{\infty} f\left(\{-w \geq \alpha\}\right) d\alpha
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For example, if $f$ is symmetric, and since $f(E) = f(\emptyset) = 0$, we have

$$\tilde{f}(-w) = \int_{-\infty}^{\infty} f\left(\{-w \geq \alpha\}\right) d\alpha = \int_{-\infty}^{\infty} f\left(\{w \leq -\alpha\}\right) d\alpha \quad (17.37)$$

Equality (a) follows since

$$\int_{-\infty}^{\infty} f(\alpha) d\alpha = \int_{-\infty}^{\infty} f(a\alpha + b) d\alpha$$

for any $b$ and $a \in \pm 1$, and equality (b) follows since $f(A) = f(E \setminus A)$, so $f(\{w \leq \alpha\}) = f(\{w > \alpha\})$. (17.39)
Lovász extension properties

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Given Eqns. (17.30) through (17.33), most of the above properties are relatively easy to derive.

For example, if \( f \) is symmetric, and since \( f(E^c) = f(\emptyset) = 0 \), we have

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Lovász extension properties

- Given Eqns. (17.30) through (17.33), most of the above properties are relatively easy to derive.
- For example, if $f$ is symmetric, and since $f(E') = f(\emptyset) = 0$, we have

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(a) $$
\int_{-\infty}^{\infty} f\left\{ w \leq \alpha \right\} d\alpha \overset{(a)}{=} \int_{-\infty}^{\infty} f\left\{ w > \alpha \right\} d\alpha
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Equality (a) follows since $\int_{-\infty}^{\infty} f(\alpha) d\alpha = \int_{-\infty}^{\infty} f(a\alpha + b) d\alpha$ for any $b$ and $a \in \pm 1$, and equality (b) follows since $f(A) = f(E \setminus A)$, so $f(\{w \leq \alpha\}) = f(\{w > \alpha\})$. 


Recall, for \( w \in \mathbb{R}^E_+ \), we have \( \tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\}) \, d\alpha \).
Recall, for \( w \in \mathbb{R}^E_+ \), we have \( \tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\}) d\alpha \).

Since \( f(\{w \geq \alpha\}) = 0 \) for \( \alpha > w_1 \geq w_\ell \), we have for \( w \in \mathbb{R}^E_+ \), we have \( \tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\}) d\alpha \).
Recall, for $w \in \mathbb{R}^E_+$, we have $\tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\}) d\alpha$

Since $f(\{w \geq \alpha\}) = 0$ for $\alpha > w_1 \geq w_\ell$, we have for $w \in \mathbb{R}^E_+$, we have $\tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\}) d\alpha$

For $w \in [0, 1]^E$, then
$\tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\}) d\alpha = \int_0^1 f(\{w \geq \alpha\}) d\alpha$ since $f(\{w \geq \alpha\}) = 0$ for $1 \geq \alpha > w_1$.
Lovász extension, expected value of random variable

- Recall, for $w \in \mathbb{R}_+^E$, we have $\tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\})d\alpha$
- Since $f(\{w \geq \alpha\}) = 0$ for $\alpha > w_1 \geq w_\ell$, we have for $w \in \mathbb{R}_+^E$, we have $\tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\})d\alpha$
- For $w \in [0, 1]^E$, then $\tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\})d\alpha = \int_0^1 f(\{w \geq \alpha\})d\alpha$ since $f(\{w \geq \alpha\}) = 0$ for $1 \geq \alpha > w_1$.
- Consider $\alpha$ as a uniform random variable on $[0, 1]$ and let $h(\alpha)$ be a function of $\alpha$. Then the expected value $\mathbb{E}[f(\alpha)] = \int_0^1 h(\alpha)d\alpha$. 
Lovász extension, expected value of random variable

- Recall, for \( w \in \mathbb{R}_+^E \), we have \( \tilde{f}(w) = \int_0^\infty f(\{w \geq \alpha\})d\alpha \)
- Since \( f(\{w \geq \alpha\}) = 0 \) for \( \alpha > w_1 \geq w_\ell \), we have for \( w \in \mathbb{R}_+^E \), we have \( \tilde{f}(w) = \int_0^{w_1} f(\{w \geq \alpha\})d\alpha \)
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- Consider \( \alpha \) as a uniform random variable on \([0, 1]\) and let \( h(\alpha) \) be a function of \( \alpha \). Then the expected value \( \mathbb{E}[f(\alpha)] = \int_0^1 h(\alpha)d\alpha \).
- Hence, for \( w \in [0, 1]^m \), we can also define the Lovász extension as
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  \tilde{f}(w) = \mathbb{E}[f(\{w \geq \alpha\})] = \mathbb{E}[f(e \in E: w(e_i) \geq \alpha)] \tag{17.40}
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This is very useful for showing results for various randomized rounding schemes when solving submodular optimization problems subject to constraints via relaxations to convex optimization problems subject to linear constraints.
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Ellipsoid algorithm, and polynomial time SFM

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**Definition 17.5.1 ((strong) optimization problem)**

Given $c \in \mathbb{R}^V$, find a vector $x \in C$ that maximizes $c^T x$ on $C$. I.e., solve

$$\max_{x \in C} c^T x$$

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**Definition 17.5.2 ((strong) separation problem)**

Given a vector $y \in \mathbb{R}^V$, decide if $y \in C$, and if not, find a hyperplane that separates $y$ from $C$. I.e., find vector $c \in \mathbb{R}^V$ such that:

$$c^T y > \max_{x \in C} c^T x$$  \hspace{1cm} (17.42)
Ellipsoid algorithm, and polynomial time SFM

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- See also, the book: Grötschel, Lovász, and Schrijver, “Geometric Algorithms and Combinatorial Optimization”
SFM is also related to the convexity of the Lovász extension, the ease of minimizing convex functions.
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Also, since we can recover $f$ from $\tilde{f}$ via $f(A) = \tilde{f}(1_A)$, and (as we will see) get discrete solutions from continuous convex minimization solution.
Minimizing $\tilde{f}$ vs. minimizing $f$

In fact, we have:

**Theorem 17.5.4**

Let $f$ be submodular and $\tilde{f}$ be its Lovász extension. Then

$$\min \{ f(A) | A \subseteq E \} = \min_{w \in \{0,1\}^E} \tilde{f}(w) = \min_{w \in [0,1]^E} \tilde{f}(w).$$
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- Also, $\sum_i \lambda_i = w(e_1) \leq 1$. 

...
Minimizing \( \tilde{f} \) vs. minimizing \( f \)

...cont. proof of Thm. 17.5.4.

- Note that since \( f(\emptyset) = 0 \), \( \min \{ f(A) | A \subseteq E \} \leq 0 \).
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\tilde{f}(w) = \int_0^1 f(\{ w \geq \alpha \}) d\alpha = \sum_{i=1}^m \lambda_i f(E_i) = (17.43)
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Thus, \( \min \{ f(A) | A \subseteq E \} = \min_{w \in [0,1]^E} \tilde{f}(w) \).
Other minimizers based on min of $\tilde{f}$

Let $w^* \in \text{argmin} \left\{ \tilde{f}(w) | w \in [0, 1]^E \right\}$ and let $A^* \in \text{argmin} \left\{ f(A) | A \subseteq V \right\}$. 

Previous theorem states that $\tilde{f}(w^*) = f(A^*)$.

Let $\lambda^*_i$ be the function weights and $E^*_i$ be the sets associated with $w^*$. From previous theorem, we have $\tilde{f}(w^*) = \sum_i \lambda^*_i f(E^*_i) = f(A^*) = \min \{ f(A) | A \subseteq E \}$. 

Thus, since $w^* \in [0, 1]^E$, each $0 \leq \lambda^*_i \leq 1$, we have for all $i$ such that $\lambda^*_i > 0$, $f(E^*_i) = f(A^*)$ (17.47) meaning such $E^*_i$ are also minimizers of $f$, and $\sum_i \lambda_i = 1$.

Note that the negative of $f(A^*)$ is crucial here. See next slide that further explains this.

Hence $w^* = \sum_i \lambda^*_i 1_{E^*_i}$ is in convex hull of incidence vectors of minimizers of $f$. 

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$$\tilde{f}(w^*) = \sum_i \lambda_i^* f(E_i^*) = f(A^*) = \min \left\{ f(A) | A \subseteq E \right\} \quad (17.46)$$
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A bit more on level sets being minimizers

- $f$ is normalized $f(\emptyset) = 0$, so minimizer is $\leq 0$. 
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- Then we have

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    f(A^*) = \sum_i \lambda_i f(E_i^*) > \sum_i \lambda_i f(A^*) = f(A^*) \sum_i \lambda_i \quad (17.48)
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and since \( f(A^*) < 0 \), this means that \( \sum_i \lambda_i > 1 \) which is a contradiction.
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$$f(A^*) = \sum_i \lambda_i f(E^*_i) > \sum_i \lambda_i f(A^*) = f(A^*) \sum_i \lambda_i \quad (17.48)$$

and since $f(A^*) < 0$, this means that $\sum_i \lambda_i > 1$ which is a contradiction.
- Hence, must have $f(E^*_i) = f(A^*)$ for all $i$.
- Hence, $\sum_i \lambda_i = 1$ since $f(A^*) = \sum_i \lambda_i f(A^*)$. 
We know \( f(A^*) \leq 0 \). Consider two cases in Equation 17.47.

Case 1: \( f(A^*) = 0 \). Then for any \( i \) with \( \lambda_i > 0 \) we must have \( f(E_i) = 0 \) as well for all \( i \) since \( f(A^*) \leq f(E_i) \).

Case 2 is where \( f(A^*) < 0 \). In this second case, we have

\[
0 > f(A^*) = \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) \tag{17.49}
\]

\[
\geq \sum_i \lambda_i f(A^*) + (1 - \bar{\lambda}) f(A^*) = f(A^*) \tag{17.50}
\]

where \( \bar{\lambda} = \sum_i \lambda_i \) and \( (1 - \bar{\lambda}) \geq 0 \) and where (a) follows since \( f(A^*) < 0 \).

Hence, all inequalities must be equalities, which means that we must have that \( \bar{\lambda} = 1 \).
Alternate way to see Equation 17.47

- We know $f(A^*) \leq 0$. Consider two cases in Equation 17.47.

  - Case 1: $f(A^*) = 0$. Then for any $i$ with $\lambda_i > 0$ we must have $f(E_i) = 0$ as well for all $i$ since $f(A^*) \leq f(E_i)$. 

- Case 2 is where $f(A^*) < 0$. In this second case, we have $0 > f(A^*) = \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) (17.49)$

  $(a)$ follows since $f(A^*) < 0$. Hence, all inequalities must be equalities, which means that we must have that $\bar{\lambda} = 1$. 

Alternate way to see Equation 17.47

- We know $f(A^*) \leq 0$. Consider two cases in Equation 17.47.
- Case 1: $f(A^*) = 0$. Then for any $i$ with $\lambda_i > 0$ we must have $f(E_i) = 0$ as well for all $i$ since $f(A^*) \leq f(E_i)$.
- Case 2 is where $f(A^*) < 0$. In this second case, we have

\begin{align*}
0 > f(A^*) &= \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) \\
&\geq \sum_i \lambda_i f(A^*) + (1 - \bar{\lambda}) f(A^*) = f(A^*)
\end{align*}

where $\bar{\lambda} = \sum_i \lambda_i$ and $(1 - \bar{\lambda}) \geq 0$ and where (a) follows since $f(A^*) < 0$. 

(a)
Alternate way to see Equation 17.47

- We know $f(A^*) \leq 0$. Consider two cases in Equation 17.47.

- Case 1: $f(A^*) = 0$. Then for any $i$ with $\lambda_i > 0$ we must have $f(E_i) = 0$ as well for all $i$ since $f(A^*) \leq f(E_i)$.

- Case 2 is where $f(A^*) < 0$. In this second case, we have

$$0 > f(A^*) = \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) \quad (17.49)$$

$$\geq \sum_i \lambda_i f(A^*) + (1 - \bar{\lambda}) f(A^*) = f(A^*) \quad (17.50)$$

where $\bar{\lambda} = \sum_i \lambda_i$ and $(1 - \bar{\lambda}) \geq 0$ and where (a) follows since $f(A^*) < 0$.

- Hence, all inequalities must be equalities, which means that we must have that $\bar{\lambda} = 1$. 
We can also view the above as a form of rounding a continuous convex relaxation to the problem.

**Definition 17.5.5 (θ-rounding)**

Given vector $x \in [0, 1]^E$, choose $\theta \in (0, 1)$ and define a set corresponding to elements above $\theta$, i.e.,

$$\hat{X}_\theta = \{ i : \hat{x}(i) \geq \theta \} \triangleq \{ \hat{x} \geq \theta \}$$  \hspace{1cm} (17.51)

**Lemma 17.5.6 (Fujishige-2005)**

*Given a continuous minimizer $x^* \in \arg\min_{x \in [0, 1]^n} \tilde{f}(x)$, the discrete minimizers are exactly the maximal chain of sets $\emptyset \subset X_{\theta_1} \subset \ldots X_{\theta_k}$ obtained by $\theta$-rounding $x^*$, for $\theta_j \in (0, 1)$.***
Simple expressions for Lovász E. with $m = 2$, $E = \{1, 2\}$

- If $w_1 \geq w_2$, then

$$
\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\}|\{1\}) \\
= (w_1 - w_2) f(\{1\}) + w_2 f(\{1, 2\})
$$

(17.52)
Simple expressions for Lovász E. with $m = 2$, $E = \{1, 2\}$

- If $w_1 \geq w_2$, then
  \[
  \tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} \mid \{1\})
  \]
  \[
  = (w_1 - w_2) f(\{1\}) + w_2 f(\{1, 2\})
  \]
  \[
  = (w_1 - w_2) f(\{1\}) + w_2 f(\{1, 2\})
  \]
  \[
  (17.52)
  \]

- If $w_1 \leq w_2$, then
  \[
  \tilde{f}(w) = w_2 f(\{2\}) + w_1 f(\{1\} \mid \{2\})
  \]
  \[
  = (w_2 - w_1) f(\{2\}) + w_1 f(\{1, 2\})
  \]
  \[
  = (w_2 - w_1) f(\{2\}) + w_1 f(\{1, 2\})
  \]
  \[
  (17.55)
  \]
If \( w_1 \geq w_2 \), then

\[
\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} \setminus \{1\}) \\
= (w_1 - w_2) f(\{1\}) + w_2 f(\{1, 2\}) \\
= \frac{1}{2} f(1)(w_1 - w_2) + \frac{1}{2} f(1)(w_1 - w_2) \\
+ \frac{1}{2} f(\{1, 2\})(w_1 + w_2) - \frac{1}{2} f(\{1, 2\})(w_1 - w_2) \\
+ \frac{1}{2} f(2)(w_1 - w_2) + \frac{1}{2} f(2)(w_2 - w_1)
\]
If \( w_1 \geq w_2 \), then

\[
\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} | \{1\}) \tag{17.56}
\]

\[
= (w_1 - w_2) f(\{1\}) + w_2 f(\{1, 2\}) \tag{17.57}
\]

\[
= \frac{1}{2} f(1)(w_1 - w_2) + \frac{1}{2} f(1)(w_1 - w_2) \tag{17.58}
\]

\[
+ \frac{1}{2} f(\{1, 2\})(w_1 + w_2) - \frac{1}{2} f(\{1, 2\})(w_1 - w_2) \tag{17.59}
\]

\[
+ \frac{1}{2} f(2)(w_1 - w_2) + \frac{1}{2} f(2)(w_2 - w_1) \tag{17.60}
\]

A similar (symmetric) expression holds when \( w_1 \leq w_2 \).
This gives, for general $w_1, w_2$, that

\[
\tilde{f}(w) = \frac{1}{2} (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) |w_1 - w_2| \quad (17.61)
\]

\[
+ \frac{1}{2} (f(\{1\}) - f(\{2\}) + f(\{1, 2\})) w_1 \quad (17.62)
\]

\[
+ \frac{1}{2} (-f(\{1\}) + f(\{2\}) + f(\{1, 2\})) w_2 \quad (17.63)
\]

\[
= - (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) \min \{w_1, w_2\} \quad (17.64)
\]

\[
+ f(\{1\}) w_1 + f(\{2\}) w_2 \quad (17.65)
\]
Simple expressions for Lovász E. with $m = 2$, $E = \{1, 2\}$

This gives, for general $w_1, w_2$, that

$$\tilde{f}(w) = \frac{1}{2} (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) |w_1 - w_2|$$  \hspace{1cm} (17.61)

$$+ \frac{1}{2} (f(\{1\}) - f(\{2\}) + f(\{1, 2\})) w_1$$  \hspace{1cm} (17.62)

$$+ \frac{1}{2} (-f(\{1\}) + f(\{2\}) + f(\{1, 2\})) w_2$$  \hspace{1cm} (17.63)

$$= - (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) \min \{w_1, w_2\}$$  \hspace{1cm} (17.64)

$$+ f(\{1\}) w_1 + f(\{2\}) w_2$$  \hspace{1cm} (17.65)

Thus, if $f(A) = H(X_A)$ is the entropy function, we have

$$\tilde{f}(w) = H(e_1) w_1 + H(e_2) w_2 - I(e_1; e_2) \min \{w_1, w_2\}$$ which must be convex in $w$, where $I(e_1; e_2)$ is the mutual information.
Simple expressions for Lovász E. with $m = 2$, $E = \{1, 2\}$

- This gives, for general $w_1, w_2$, that

  \[
  \tilde{f}(w) = \frac{1}{2} (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) |w_1 - w_2| \\
  + \frac{1}{2} (f(\{1\}) - f(\{2\}) + f(\{1, 2\})) w_1 \\
  + \frac{1}{2} (-f(\{1\}) + f(\{2\}) + f(\{1, 2\})) w_2 \\
  = - (f(\{1\}) + f(\{2\}) - f(\{1, 2\})) \min \{w_1, w_2\} \\
  + f(\{1\}) w_1 + f(\{2\}) w_2
  \]

- Thus, if $f(A) = H(X_A)$ is the entropy function, we have

  \[
  \tilde{f}(w) = H(e_1) w_1 + H(e_2) w_2 - I(e_1; e_2) \min \{w_1, w_2\}
  \]

  which must be convex in $w$, where $I(e_1; e_2)$ is the mutual information.

- This “simple” but general form of the Lovász extension with $m = 2$ can be useful.
Example: \( m = 2, \ E = \{1, 2\}, \) contours

- If \( w_1 \geq w_2, \) then

\[
\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\}|\{1\}) \tag{17.66}
\]
Example: $m = 2$, $E = \{1, 2\}$, contours

- If $w_1 \geq w_2$, then
  \[
  \tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} | \{1\})
  \]  \hfill (17.66)

- If $w = (1, 0)/f(\{1\}) = \left(1/f(\{1\}), 0\right)$ then $\tilde{f}(w) = 1$. 

Example: \( m = 2, \ E = \{1, 2\}, \) contours

- If \( w_1 \geq w_2, \) then

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\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\}|\{1\})
\]

(17.66)

- If \( w = (1, 0)/f(\{1\}) = \left(1/f(\{1\}), 0\right) \) then \( \tilde{f}(w) = 1. \)
- If \( w = (1, 1)/f(\{1, 2\}) \) then \( \tilde{f}(w) = 1. \)
Example: \( m = 2, \ E = \{1, 2\}, \) contours

- If \( w_1 \geq w_2, \) then
  \[
  \tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\}|\{1\}) \tag{17.66}
  \]

  - If \( w = (1, 0)/f(\{1\}) = (1/f(\{1\}), 0) \) then \( \tilde{f}(w) = 1. \)
  - If \( w = (1, 1)/f(\{1, 2\}) \) then \( \tilde{f}(w) = 1. \)

- If \( w_1 \leq w_2, \) then
  \[
  \tilde{f}(w) = w_2 f(\{2\}) + w_1 f(\{1\}|\{2\}) \tag{17.67}
  \]
Example: \( m = 2, \ E = \{1, 2\}, \) contours

- If \( w_1 \geq w_2, \) then
  \[
  \tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} | \{1\})
  \]
  (17.66)

  - If \( w = (1, 0) / f(\{1\}) = (1 / f(\{1\}), 0) \), then \( \tilde{f}(w) = 1 \).
  - If \( w = (1, 1) / f(\{1, 2\}) \), then \( \tilde{f}(w) = 1 \).

- If \( w_1 \leq w_2, \) then
  \[
  \tilde{f}(w) = w_2 f(\{2\}) + w_1 f(\{1\} | \{2\})
  \]
  (17.67)

  - If \( w = (0, 1) / f(\{2\}) = (0, 1 / f(\{2\})) \), then \( \tilde{f}(w) = 1 \).
Example: \( m = 2, \ E = \{1, 2\} \), contours

- If \( w_1 \geq w_2 \), then

\[
\tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\}|\{1\})
\]  

(17.66)

- If \( w = (1, 0)/f(\{1\}) = (1/f(\{1\}), 0) \) then \( \tilde{f}(w) = 1 \).
- If \( w = (1, 1)/f(\{1, 2\}) \) then \( \tilde{f}(w) = 1 \).

- If \( w_1 \leq w_2 \), then

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\tilde{f}(w) = w_2 f(\{2\}) + w_1 f(\{1\}|\{2\})
\]  

(17.67)

- If \( w = (0, 1)/f(\{2\}) = (0, 1/f(\{2\})) \) then \( \tilde{f}(w) = 1 \).
- If \( w = (1, 1)/f(\{1, 2\}) \) then \( \tilde{f}(w) = 1 \).
Example: $m = 2$, $E = \{1, 2\}$, contours

- If $w_1 \geq w_2$, then
  \[
  \tilde{f}(w) = w_1 f(\{1\}) + w_2 f(\{2\} \mid \{1\})
  \]  \hspace{1cm} (17.66)

  - If $w = (1, 0)/f(\{1\}) = \left(1/f(\{1\}), 0\right)$ then $\tilde{f}(w) = 1$.
  - If $w = (1, 1)/f(\{1, 2\})$ then $\tilde{f}(w) = 1$.

- If $w_1 \leq w_2$, then
  \[
  \tilde{f}(w) = w_2 f(\{2\}) + w_1 f(\{1\} \mid \{2\})
  \]  \hspace{1cm} (17.67)

  - If $w = (0, 1)/f(\{2\}) = (0, 1/f(\{2\}))$ then $\tilde{f}(w) = 1$.
  - If $w = (1, 1)/f(\{1, 2\})$ then $\tilde{f}(w) = 1$.

- Can plot contours of the form $\left\{ w \in \mathbb{R}^2 : \tilde{f}(w) = 1 \right\}$, particular marked points of form $w = 1_A \times \frac{1}{f(A)}$ for certain $A$, where $\tilde{f}(w) = 1$. 

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Prof. Jeff Bilmes
Example: \( m = 2, \ E = \{1, 2\} \)

- Contour plot of \( m = 2 \) Lovász extension (from Bach-2011).

\[
\{ w : f(\tilde{w}) = 1 \}
\]

\[
(0, 1) / f(\{2\}) \quad (1, 1) / f(\{1, 2\})
\]

\[
(1, 0) / f(\{1\})
\]
Example: $m = 3$, $E = \{1, 2, 3\}$

- In order to visualize in 3D, we make a few simplifications.
Example: $m = 3, E = \{1, 2, 3\}$

- In order to visualize in 3D, we make a few simplifications.
- Consider any submodular $f'$ and $x \in B_{f'}$. Then $f(A) = f'(A) - x(A)$ is submodular
Example: $m = 3, \ E = \{1, 2, 3\}$

- In order to visualize in 3D, we make a few simplifications.
- Consider any submodular $f'$ and $x \in B_{f'}$. Then $f(A) = f'(A) - x(A)$ is submodular, and moreover $f(E) = f'(E) - x(E) = 0$. 
Example: \( m = 3, \ E = \{1, 2, 3\} \)

- In order to visualize in 3D, we make a few simplifications.
- Consider any submodular \( f' \) and \( x \in B_{f'} \). Then 
  \[ f(A) = f'(A) - x(A) \]
  is submodular, and moreover 
  \[ f(E) = f'(E) - x(E) = 0. \]
- Hence, from \( \tilde{f}(w + \alpha 1_E) = \tilde{f}(w) + \alpha f(E) \), we have that 
  \[ \tilde{f}(w + \alpha 1_E) = \tilde{f}(w). \]
Example: \( m = 3, \ E = \{1, 2, 3\} \)

- In order to visualize in 3D, we make a few simplifications.
- Consider any submodular \( f' \) and \( x \in B_{f'} \). Then 
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  \]
  is submodular, and moreover
  \[
  f(E) = f'(E) - x(E) = 0.
  \]
- Hence, from \( \tilde{f}(w + \alpha 1_E) = \tilde{f}(w) + \alpha f(E) \), we have that
  \[
  \tilde{f}(w + \alpha 1_E) = \tilde{f}(w).
  \]
- Thus, we can look “down” on the contour plot of the Lovász extension, \( \{w : \tilde{f}(w) = 1\} \), from a vantage point right on the line
  \[
  \{x : x = \alpha 1_E, \alpha > 0\}
  \]
  since moving in direction \( 1_E \) changes nothing.
Example: \( m = 3, \ E = \{1, 2, 3\} \)

- Example 1 (from Bach-2011): 
  \[ f(A) = 1_{|A| \in \{1, 2\}} = \min\{|A|, 1\} + \min\{|E \setminus A|, 1\} - 1 \]
  is submodular, and
  \[ \tilde{f}(w) = \max_{k \in \{1, 2, 3\}} w_k - \min_{k \in \{1, 2, 3\}} w_k. \]
Example: \( m = 3, \ E = \{1, 2, 3\} \)

- Example 1 (from Bach-2011):
  \[
  f(A) = \mathbf{1}_{|A| \in \{1, 2\}} = \min \{|A|, 1\} + \min \{|E \setminus A|, 1\} - 1
  \]
  is submodular, and
  \[
  \tilde{f}(w) = \max_{k \in \{1, 2, 3\}} w_k - \min_{k \in \{1, 2, 3\}} w_k.
  \]
Example: $m = 3$, $E = \{1, 2, 3\}$

Example 2 (from Bach-2011):

$$f(A) = |\mathbf{1}_{1 \in A} - \mathbf{1}_{2 \in A}| + |\mathbf{1}_{2 \in A} - \mathbf{1}_{3 \in A}|$$

This gives a "total variation" function for the Lovász extension, with

$$\tilde{f}(w) = |w_1 - w_2| + |w_2 - w_3|$$

The graph shows the points $(0,0,1)$, $(0,1,1)$, $(1,0,1)/2$, $(0,1,0)/2$, $(1,0,0)$, and $(1,1,0)$.
Example: $m = 3, \ E = \{1, 2, 3\}$

- Example 2 (from Bach-2011):
  $$f(A) = |1_{1 \in A} - 1_{2 \in A}| + |1_{2 \in A} - 1_{3 \in A}|$$

- This gives a “total variation” function for the Lovász extension, with
  $$\tilde{f}(w) = |w_1 - w_2| + |w_2 - w_3|,$$
  a prior to prefer piecewise-constant signals.
Total Variation Example

From “Nonlinear total variation based noise removal algorithms” Rudin, Osher, and Fatemi, 1992. Top left original, bottom right total variation.
Example: Lovász extension of concave over modular

Let \( m : E \to \mathbb{R}_+ \) be a modular function and define
\[
f(A) = g(m(A))
\]
where \( g \) is concave. Then \( f \) is submodular.
Example: Lovász extension of concave over modular

- Let $m : E \rightarrow \mathbb{R}_+$ be a modular function and define $f(A) = g(m(A))$ where $g$ is concave. Then $f$ is submodular.
- Let $M_j = \sum_{i=1}^{j} m(e_i)$
Example: Lovász extension of concave over modular

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  where \( g \) is concave. Then \( f \) is submodular.
- Let \( M_j = \sum_{i=1}^{j} m(e_i) \)
- \( \tilde{f}(w) \) is given as

\[
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i) \left( g(M_i) - g(M_{i-1}) \right) \tag{17.68}
\]
Example: Lovász extension of concave over modular

- Let $m : E \rightarrow \mathbb{R}_+$ be a modular function and define $f(A) = g(m(A))$ where $g$ is concave. Then $f$ is submodular.
- Let $M_j = \sum_{i=1}^{j} m(e_i)$
- $\tilde{f}(w)$ is given as

$$
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i)(g(M_i) - g(M_{i-1})) \quad (17.68)
$$

- And if $m(A) = |A|$, we get

$$
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i)(g(i) - g(i - 1)) \quad (17.69)
$$
Example: Lovász extension and cut functions

- Cut Function: Given a non-negative weighted graph $G = (V, E, m)$ where $m : E \rightarrow \mathbb{R}_+$ is a modular function over the edges, we know from Lecture 2 that $f : 2^V \rightarrow \mathbb{R}_+$ with $f(X) = m(\Gamma(X))$ where $\Gamma(X) = \{(u, v) | (u, v) \in E, u \in X, v \in V \setminus X\}$ is non-monotone submodular.
Example: Lovász extension and cut functions

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- **Simple way to write it**, with $m_{ij} = m((i, j))$:

  $$f(X) = \sum_{i \in X, j \in V \setminus X} m_{ij} \quad (17.70)$$

Exercise: show that Lovász extension of graph cut may be written as:

$$\tilde{f}(w) = \sum_{i, j \in V} m_{ij} \max\{w_i - w_j, 0\} \quad (17.71)$$

where elements are ordered as usual, $w_1 \geq w_2 \geq \cdots \geq w_n$. This is also a form of "total variation".
Example: Lovász extension and cut functions

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  \[
  \tilde{f}(w) = \sum_{i, j \in V} m_{ij} \max\{(w_i - w_j), 0\} \quad (17.71)
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Example: Lovász extension and cut functions

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- Simple way to write it, with \( m_{ij} = m((i, j)) \):
  \[
f(X) = \sum_{i \in X, j \in V \setminus X} m_{ij} \quad (17.70)
  \]

- **Exercise:** show that Lovász extension of graph cut may be written as:
  \[
  \tilde{f}(w) = \sum_{i, j \in V} m_{ij} \max \{(w_i - w_j), 0\} \quad (17.71)
  \]
  where elements are ordered as usual, \( w_1 \geq w_2 \geq \cdots \geq w_n \).

- **This is also a form of “total variation”**
Some additional submodular functions and their Lovász extensions, where \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \geq 0 \). Let \( W_k \triangleq \sum_{i=1}^{k} w(e_i) \).

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(thanks to K. Narayanan).
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- Given training data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^m$ with $(x_i, y_i) \in \mathbb{R}^n \times \mathbb{R}$, perform the following risk minimization problem:

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- When data has multiple responses only that are observed, \((y_i) \in \mathbb{R}^k\) we get dictionary learning (Krause & Guestrin, Das & Kempe):

\[
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Norms, sparse norms, and computer vision

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- Points of difference should be “sparse” (frequently zero).

(Rodriguez, 2009)
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- Ex: total variation is Lovász-ext. of graph cut, but $\exists$ many more!
Lovász extension and norms

- Using Lovász extension to define various norms of the form $\|w\|_\tilde{f} = \tilde{f}(|w|)$, renders the function $f$ symmetric about all orthants (i.e., $\|w\|_\tilde{f} = \|b \odot w\|_\tilde{f}$ where $b \in \{-1, 1\}^m$ and $\odot$ is element-wise multiplication).
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- Bach-2011 has a complete discussion of this.