Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 5 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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April 14th, 2014
Read chapter 1 from Fujishige’s book.
our room (Mueller Hall Room 154) is changed!

Please do use our discussion board (https://canvas.uw.edu/courses/895956/discussion_topics) for all questions, comments, so that all will benefit from them being answered.

Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).
Class Road Map - IT-I

L1 (3/31): Motivation, Applications, & Basic Definitions
L2: (4/2): Applications, Basic Definitions, Properties
L3: More examples and properties (e.g., closure properties), and examples, spanning trees
L4: proofs of equivalent definitions, independence, start matroids
L5: matroids, basic definitions and examples
L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
L7: Dual Matroids, other matroid properties, Combinatorial Geometries
L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
L9:
L10:
L11:
L12:
L13:
L14:
L15:
L16:
L17:
L18:
L19:
L20:

Finals Week: June 9th-13th, 2014.
Many (Equivalent) Definitions of Submodularity

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \ \forall A, B \subseteq V \] (5.6)

\[ f(j|S) \geq f(j|T), \ \forall S \subseteq T \subseteq V, \ \text{with} \ j \in V \setminus T \] (5.7)

\[ f(C|S) \geq f(C|T), \ \forall S \subseteq T \subseteq V, \ \text{with} \ C \subseteq V \setminus T \] (5.8)

\[ f(j|S) \geq f(j|S \cup \{k\}), \ \forall S \subseteq V \ \text{with} \ j \in V \setminus (S \cup \{k\}) \] (5.9)

\[ f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \ \forall A, B \subseteq V \] (5.10)

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \ \forall S, T \subseteq V \] (5.11)

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \ \forall S \subseteq T \subseteq V \] (5.12)

\[ f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T), \ \forall S, T \subseteq V \] (5.13)
We saw: column space of a matrix, dimensionality of span of subset of columns as rank function.
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Incidence matrix of (arbitrarily oriented version of) graph $G = (V, E)$, rank of matrix columns $F$ corresponded to spanning tree of edge-induced graph $G' = (V', F)$ where $v'$ are vertices incident to edges in $F$. 

We wish to more formally connect the above, and generalize further. 

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EE596b/Spring 2014/Submodularity - Lecture 5 - April 14th, 2014
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We saw several different “greedy” algorithms that proceed optimal spanning trees (Borůvka’s, Jarník/Prim/Dijkstra’s, and Kruskal’s).
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We wish to more formally connect the above, and generalize further.
So $V$ is set of column vector indices of a matrix.
Let $\mathcal{I}$ be a set of all subsets of $V$ such that for any $I \in \mathcal{I}$, the vectors indexed by $I$ are linearly independent.
Given a set $B \in \mathcal{I}$ of linearly independent vectors, then any subset $A \subseteq B$ is also linearly independent. Hence, $\mathcal{I}$ is down-closed or "subclusive", under subsets. In other words,

$$A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I} \tag{5.32}$$

maxInd: Inclusionwise maximal independent subsets (or bases) of any set $B \subseteq V$.

$$\text{maxInd}(B) \triangleq \{ A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I} \} \tag{5.33}$$

Given any set $B \subseteq V$ of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all $B \subseteq V$,

$$\forall A_1, A_2 \in \text{maxInd}(B), \quad |A_1| = |A_2| \tag{5.34}$$
Thus, for all $I \in \mathcal{I}$, the matrix rank function has the property

$$r(I) = |I|$$  \hspace{1cm} (5.32)

and for any $B \notin \mathcal{I}$,

$$r(B) = \max \{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} \leq |B|$$  \hspace{1cm} (5.33)
Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then $J$ is said to be an independent set.

**Definition 5.2.4 (Matroid)**

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)? Because could have an (albeit trivial) matroid where $\mathcal{I} = \{\emptyset\}$. 
On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.
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- Understanding matroids crucial for understanding submodularity.
Matroids

Matroid Examples

Matroid Rank

Partition Matroid

On Matroids

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- Matroid independent sets (i.e., \( A \) s.t. \( r(A) = |A| \)) are useful constraint set, and fast algorithms for submodular optimization subject to one (or more) matroid independence constraints exist.
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- Matroid independent sets (i.e., $A$ s.t. $r(A) = |A|$) are useful constraint set, and fast algorithms for submodular optimization subject to one (or more) matroid independence constraints exist.
- Crapo & Rota preferred the term “combinatorial geometry”, or more specifically a “pregeometry” and said that pregeometries are “often described by the ineffably cacaphonic [sic] term ’matroid’, which we prefer to avoid in favor of the term ’pregeometry’.”
Slight modification (non unit increment) that is equivalent.

Definition 5.3.1 (Matroid-II)

A set system $(E, \mathcal{I})$ is a Matroid if

1. $\emptyset \in \mathcal{I}$ (I1')
2. $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (or “down-closed”) (I2')
3. $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$ (I3')

Note (I1)≡(I1'), (I2)≡(I2'), and we get (I3)≡(I3') using induction.
Matroids, independent sets, and bases

- **Independent sets:** Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise $A$ is called dependent.
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- **A base of $U \subseteq E$**: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$. 
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- **A base of a matroid:** If $U = E$, then a “base of $E$” is just called a base of the matroid $M$ (this corresponds to a basis in a linear space).
Proposition 5.3.2

In a matroid $M = (E, I)$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.
Matroids - important property

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- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
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- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.
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Definition 5.3.3 (Matroid)

A set system \( (V, I) \) is a Matroid if
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Definition 5.3.3 (Matroid)

A set system $(V, I)$ is a Matroid if

(I1’) $\emptyset \in I$ (emptyset containing)
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**Proposition 5.3.2**

*In a matroid* $M = (E, \mathcal{I})$, *for any* $U \subseteq E(M)$, *any two bases of* $U$ *have the same size.*

- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). **Exercise:** show the following is equivalent to the above.

**Definition 5.3.3 (Matroid)**

A set system $(V, \mathcal{I})$ is a Matroid if

(I1′) $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2′) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)
Matroids - important property

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- (I1′) \( \emptyset \in \mathcal{I} \) (emptyset containing)
- (I2′) \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \) (down-closed or subclusive)
- (I3′) \( \forall X \subseteq V \), and \( I_1, I_2 \in \text{maxInd}(X) \), we have \( |I_1| = |I_2| \) (all maximally independent subsets of \( X \) have the same size).
Matroids - rank

Thus, in any matroid $M = (E, I)$, $\forall U \subseteq E(M)$, any two bases of $U$ have the same size.
Matroids - rank

- Thus, in any matroid $M = (E, \mathcal{I})$, $\forall U \subseteq E(M)$, any two bases of $U$ have the same size.
- The common size of all the bases of $U$ is called the rank of $U$, denoted $r_M(U)$ or just $r(U)$ when the matroid in question is unambiguous.
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$r(E) = r(E, I)$ is the rank of the matroid, and is the common size of all the bases of the matroid.
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- $r(E) = r_{(E,I)}$ is the rank of the matroid, and is the common size of all the bases of the matroid.
- We can a bit more formally define the rank function this way.
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**Definition 5.3.4 (matroid rank function)**

The rank of a matroid is a function $r : 2^E \rightarrow \mathbb{Z}_+$ defined by

$$ r(A) = \max \{|X| : X \subseteq A, X \in I\} = \max_{X \in I} |A \cap X| \quad (5.1) $$
Thus, in any matroid $M = (E, \mathcal{I})$, $\forall U \subseteq E(M)$, any two bases of $U$ have the same size.

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From the above, we immediately see that $r(A) \leq |A|$.
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From the above, we immediately see that $r(A) \leq |A|$.

Moreover, if $r(A) = |A|$, then $A \in \mathcal{I}$, meaning $A$ is independent (in this case, $A$ is a self base).
Matroids, other definitions using matroid rank $r : 2^V \rightarrow \mathbb{Z}_+$

**Definition 5.3.5 (closed/flat/subspace)**

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$. 

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Definition 5.3.6 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.
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Therefore, a closed set $A$ has $\text{span}(A) = A$. 
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**Definition 5.3.7 (circuit)**

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 5.3.8 (Matroid (by bases))**

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
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Properties 2 and 3 are called “exchange properties.”
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 5.3.9 (Matroid by circuits)**

Let $E$ be a set and $\mathcal{C}$ be a collection of subsets of $E$ that satisfy the following three properties:

1. (C1): $\emptyset \notin \mathcal{C}$
2. (C2): if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
3. (C3): if $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.
Several circuit definitions for matroids.

**Theorem 5.3.10 (Matroid by circuits)**

Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.

1. $C$ is the collection of circuits of a matroid;
2. if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;
Matroids by circuits

Several circuit definitions for matroids.

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3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.
Theorem 5.3.11 (Matroid by submodular functions)

Let \( f : 2^E \to \mathbb{Z} \) be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

\[
C(f) = \left\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \right\}
\] (5.2)

Then \( C(f) \) is the collection of circuits of a matroid on \( E \).

Inclusionwise-minimal in this case means that if \( C \in C(f) \), then there exists no \( C' \subset C \) with \( C' \in C(f) \) (i.e., \( C' \subset C \) would either be empty or have \( f(C') \geq |C'| \)). Also, recall inclusionwise-minimal in Definition 5.3.7, the definition of a circuit.
Uniform Matroid

Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{ A \subseteq E : |A| \leq k \}$.
- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \not\in I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$. 

Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
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- Rank function

$$r(A) = \begin{cases} 
|A| & \text{if } |A| \leq k \\
 k & \text{if } |A| > k
\end{cases} \quad (5.3)$$
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
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- Rank function

$$r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases}$$ (5.3)

- Note, this function is submodular. Not surprising since $r(A) = \min(|A|, k)$ which is a non-decreasing concave function applied to a modular function.
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$.
  That is $\mathcal{I} = \{ A \subseteq E : |A| \leq k \}$.
- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \notin I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$.
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  $$r(A) = \begin{cases} 
  |A| & \text{if } |A| \leq k \\
  k & \text{if } |A| > k 
  \end{cases} \quad (5.3)$$

- Note, this function is submodular. Not surprising since $r(A) = \min(|A|, k)$ which is a non-decreasing concave function applied to a modular function.
- Closure function

  $$\text{span}(A) = \begin{cases} 
  A & \text{if } |A| < k, \\
  E & \text{if } |A| \geq k, 
  \end{cases} \quad (5.4)$$
Uniform Matroid

- Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \notin I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$.
- Rank function

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- Note, this function is submodular. Not surprising since $r(A) = \min(|A|, k)$ which is a non-decreasing concave function applied to a modular function.
- Closure function

$$\text{span}(A) = \begin{cases} A & \text{if } |A| < k, \\ E & \text{if } |A| \geq k, \end{cases} \quad (5.4)$$

- A “free” matroid sets $k = |E|$, so everything is independent.
Linear (or Matric) Matroid

- Let \( X \) be an \( n \times m \) matrix and \( E = \{1, \ldots, m\} \)
Linear (or Matric) Matroid

- Let \( \mathbf{X} \) be an \( n \times m \) matrix and \( E = \{1, \ldots, m\} \)
- Let \( \mathcal{I} \) consists of subsets of \( E \) such that if \( A \in \mathcal{I} \), and \( A = \{a_1, a_2, \ldots, a_k\} \) then the vectors \( x_{a_1}, x_{a_2}, \ldots, x_{a_k} \) are linearly independent.
Linear (or Matric) Matroid

- Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
- The rank function is just the rank of the space spanned by the corresponding set of vectors.
Linear (or Matric) Matroid

- Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
- the rank function is just the rank of the space spanned by the corresponding set of vectors.
- rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$

Let $I$ consists of subsets of $E$ such that if $A \in I$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.

the rank function is just the rank of the space spanned by the corresponding set of vectors.

rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).

Called both linear matroids and matric matroids.
Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.
Let $G = (V, E)$ be a graph. Consider $(E, I)$ where the edges of the graph $E$ are the ground set and $A \in I$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, I)$ is a matroid.
Let $G = (V, E)$ be a graph. Consider $(E, I)$ where the edges of the graph $E$ are the ground set and $A \in I$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, I)$ is a matroid.

$I$ contains all forests.
Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, \mathcal{I})$ is a matroid.

$\mathcal{I}$ contains all forests.

Bases are spanning forests (spanning trees if $G$ is connected).
Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

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$\mathcal{I}$ contains all forests.

Bases are spanning forests (spanning trees if $G$ is connected).

Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$. 
Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, \mathcal{I})$ is a matroid.

$I$ contains all forests.

Bases are spanning forests (spanning trees if $G$ is connected).

Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$.

Closure function adds all edges between the vertices adjacent to any edge in $A$. Closure of a spanning forest is $G$. 
Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.
Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.
Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.
Example: graphic matroid

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A graph defines a matroid on edge sets, independent sets are those without a cycle.
Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.
Partition Matroid

- Let $V$ be our ground set.
Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}.$$ 

(5.5)

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
Let $V$ be our ground set.

Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into blocks or disjoint sets (disjoint union). Define a set of subsets of $V$ as

$$I = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \quad (5.5)$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.

Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$. 

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Partition Matroid

Let \( V \) be our ground set.

Let \( V = V_1 \cup V_2 \cup \cdots \cup V_\ell \) be a partition of \( V \) into blocks or disjoint sets (disjoint union). Define a set of subsets of \( V \) as

\[
\mathcal{I} = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \tag{5.5}
\]

where \( k_1, \ldots, k_\ell \) are fixed parameters, \( k_i \geq 0 \). Then \( M = (V, \mathcal{I}) \) is a matroid.

Note that a \( k \)-uniform matroid is a trivial example of a partition matroid with \( \ell = 1, V_1 = V \), and \( k_1 = k \).

We’ll show that property (I3’) in Def 5.3.3 holds. If \( X, Y \in \mathcal{I} \) with \( |Y| > |X| \), then there must be at least one \( i \) with \( |Y \cap V_i| > |X \cap V_i| \). Therefore, adding one element \( e \in V_i \cap (Y \setminus X) \) to \( X \) won’t break independence.
Ground set of objects, \( V = \{ \} \)
Partition Matroid

Partition of $V$ into six blocks, $V_1, V_2, \ldots, V_6$
Partition Matroid

Limit associated with each block, \( \{k_1, k_2, \ldots, k_6\} \)
Partition Matroid

Independent subset but not maximally independent.
Partition Matroid

Maximally independent subset, what is called a base.
Partition Matroid

Not independent since over limit in set six.
Matroids - rank

Lemma 5.5.1

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
 r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]
Lemma 5.5.1

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \)
**Lemma 5.5.1**

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is

\[ r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \]

**Proof.**

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \)

2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \). (We can find such a \( Y \supseteq X \) because, starting from \( X \subseteq A \cup B \), and since \( |Y| \geq |X| \), we can choose a \( y \in Y \subseteq A \cup B \) such that \( X + y \in \mathcal{I} \) but since \( y \in A \cup B \), also \( X + y \in A \cup B \). We can keep doing this while \( |Y| > |X| \) since this is a matroid.)
Lemma 5.5.1

The rank function $r : 2^E \rightarrow \mathbb{Z}_+$ of a matroid is submodular, that is
\[ r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \]

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$.
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$. 

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Lemma 5.5.1

The rank function $r : 2^E \rightarrow \mathbb{Z}_+$ of a matroid is submodular, that is

$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have

$$r(A) + r(B) \geq |X| + |Y| = r(A \cap B) + r(A \cup B)$$

(5.6)
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The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is

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$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B|$$

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Lemma 5.5.1

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is \( r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \)

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1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \)
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3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have

\[
    r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{5.6}
\]

\[
    = |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{5.7}
\]
Lemma 5.5.1

The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
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Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
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4. Then we have
\[
  r(A) + r(B) \geq |Y \cap A| + |Y \cap B|
  = |Y \cap (A \cap B)| + |Y \cap (A \cup B)|
  \geq |X| + |Y| = r(A \cap B) + r(A \cup B)
\]
Matroids

In fact, we can use the rank of a matroid for its definition.

**Theorem 5.5.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

1. **(R1)** $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
2. **(R2)** $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. **(R3)** $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- A matroid is sometimes given as $(E, r)$ where $E$ is ground set and $r$ is rank function.
In fact, we can use the rank of a matroid for its definition.

**Theorem 5.5.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
(R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$. 

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Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.
Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $I = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, I)$ is a matroid.
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$. 

...
## Proof of Theorem 5.5.2 (matroid from rank)

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- Next, assume we have (R1), (R2), and (R3). Define $I = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, I)$ is a matroid.

- First, $\emptyset \in I$.

- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank).

Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

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First, $\emptyset \in \mathcal{I}$.

Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) \quad (5.9)$$
Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid \( M = (E, I) \), we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define \( I = \{ X \subseteq E : r(X) = |X| \} \). We will show that \((E, I)\) is a matroid.
- First, \( \emptyset \in I \).
- Also, if \( Y \in I \) and \( X \subseteq Y \) then by submodularity,

\[
r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset)
\]  

(5.9)
Proof of Theorem 5.5.2 (matroid from rank).

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- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

\[
r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \geq |Y| - |Y \setminus X|
\]  

(5.9)  

(5.10)

...
Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

  $$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \geq |Y| - |Y \setminus X| = |X|$$ (5.9) (5.10) (5.11)
Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid \( M = (E, \mathcal{I}) \), we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define \( \mathcal{I} = \{ X \subseteq E : r(X) = |X| \} \). We will show that \((E, \mathcal{I})\) is a matroid.

- First, \( \emptyset \in \mathcal{I} \).

- Also, if \( Y \in \mathcal{I} \) and \( X \subseteq Y \) then by submodularity,

\[
\begin{align*}
r(X) &\geq r(Y) - r(Y \setminus X) - r(\emptyset) \\
&\geq |Y| - |Y \setminus X| \\
&= |X|
\end{align*}
\]

(5.9) (5.10) (5.11)

implying \( r(X) = |X| \), and thus \( X \in \mathcal{I} \).
Proof of Theorem 5.5.2 (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\[
    r(B) \leq r(A \cup B) \tag{5.12}
\]
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( k \leq |B| \)).

- Suppose, to the contrary, that \( \forall b \in B \setminus A, A + b \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| \). Then

\[
\begin{align*}
  r(B) &\leq r(A \cup B) & (5.12) \\
  &\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) & (5.13)
\end{align*}
\]
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

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  &= r(A \cup (B \setminus \{b_1\}))
\end{align*}
\]
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\[
 r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) = r(A \cup (B \setminus \{b_1\})) \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A)
\]
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

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Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

$$r(B) \leq r(A \cup B) \tag{5.12}$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \tag{5.13}$$

$$= r(A \cup (B \setminus \{b_1\})) \tag{5.14}$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \tag{5.15}$$

$$= r(A \cup (B \setminus \{b_1, b_2\})) \tag{5.16}$$

$$\leq \ldots \leq r(A) = |A| < |B| \tag{5.17}$$

This gives a contradiction since $B \in \mathcal{I}$. 

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Proof of Theorem 5.5.2 (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \(|A| < |B|\), so \( r(A) = |A| < r(B) = |B|\). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( k \leq |B|\)).

Suppose, to the contrary, that \( \forall b \in B \setminus A, A + b \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| \). Then

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\begin{align*}
r(B) &\leq r(A \cup B) \\
&\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
&= r(A \cup (B \setminus \{b_1\})) \\
&\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \\
&= r(A \cup (B \setminus \{b_1, b_2\})) \\
&\leq \ldots \leq r(A) = |A| < |B|
\end{align*}
\]  

giving a contradiction since \( B \in \mathcal{I} \).
Matroids from rank II

Another way of using function $r$ to define a matroid.

**Theorem 5.5.3 (Matroid from rank II)**

*Let $E$ be a finite set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$.*
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
Summarizing: Many ways to define a Matroid

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- Closure axioms (we didn’t see this, but it is possible)
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
Maximization problems for matroids

- Given a matroid \( M = (E, \mathcal{I}) \) and a modular cost function \( c : E \to \mathbb{R} \), the task is to find an \( X \in \mathcal{I} \) such that
  \[ c(X) = \sum_{x \in X} c(x) \]
  is maximum.
- This seems remarkably similar to the max spanning tree problem.
Minimization problems for matroids

- Given a matroid $M = (E, I)$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.

- This sounds like a set cover problem (find the minimum cost covering set of sets).
What is the partition matroid’s rank function?

\[
 r(A) = \ell \sum_{i=1}^{k} \min(|A \cap V_i|, k_i) \tag{5.18}
\]

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
3. Sums of submodular functions are submodular.

\(r(A)\) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
What is the partition matroid’s rank function?

A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  \hspace{1cm} (5.18)

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3. Sums of submodular functions are submodular.

$r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
Thus, we can define a matroid as $M = (V, r)$ where $r$ satisfies matroid rank axioms.

Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+^+$ with $a > b$, and any set $R \subseteq V$ with $|R| = a$, two-block partition $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$, define:

$$r(A) = \min(|A \cap R|, b) + \min(|A \cap \bar{R}|, |\bar{R}|)$$

$$= \min(|A \cap R|, b) + |A \cap \bar{R}|$$

Partition matroid figure showing this:
Truncated Matroid Rank Function

Can use this to define a truncated matroid rank function. With
\( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

\[
f_R(A) = \min \{ r(A), a \} \tag{5.21}
\]

\[
= \min \{ \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a \} \tag{5.22}
\]

\[
= \min \{ |A|, b + |A \cap \bar{R}|, a \} \tag{5.23}
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\]
Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with
\[
\mathcal{I} = \{ I \subseteq V : |I| \leq a \text{ and } |I \cap R| \leq b \},
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\[
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\]

Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).
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- For any \( B \) with \( |B \cap R| \leq b \), \( f_R(B) = a \).
Truncated Matroid Rank Function

- Can use this to define a truncated matroid rank function. With $r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|$, $b < a$, define:

$$f_R(A) = \min \{ r(A), a \}$$  \hspace{1cm} (5.21)

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- Useful for showing hardness of constrained submodular minimization. Consider sets $B \subseteq V$ with $|B| = a$.
- For $R$, we have $f_R(R) = b < a$.
- For any $B$ with $|B \cap R| \leq b$, $f_R(B) = a$.
- For any $B$ with $|B \cap R| = \ell$, with $b < \ell < a$, $f_R(B) = b + a - \ell$. 
Truncated Matroid Rank Function

Can use this to define a truncated matroid rank function. With \( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

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(5.21)  (5.22)  (5.23)

Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with

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Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).

- For \( R \), we have \( f_R(R) = b < a \).
- For any \( B \) with \( |B \cap R| \leq b \), \( f_R(B) = a \).
- For any \( B \) with \( |B \cap R| = \ell \), with \( b < \ell < a \), \( f_R(B) = b + a - \ell \).
- \( R \), the set with minimum valuation amongst size-\( a \) sets, is hidden within an exponentially larger set of size-\( a \) sets with larger valuation.
Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting $V$ denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where $V$ is the ground set, $I$ is a set of “indices”, and $E$ is the set of edges.
- $I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into $\ell$ clusters, where there are $k_i$ nodes in the $i^{th}$ group $I_i$.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$. 
Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$
### Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

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Example where $\ell = 5$,

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Example where $\ell = 5$,
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Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.

For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i)$ is the maximum matching involving $X$. 

\[ I_1 \quad I_2 \quad I_3 \quad I_4 \quad I_5 \]
\[ V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5 \]
Laminar Matroid

- We can define a matroid with structures richer than just partitions.
Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system $(V, \mathcal{F})$ is called a laminar family if for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \cap B$, $A \setminus B$, or $B \setminus A$ is empty.
We can define a matroid with structures richer than just partitions.

A set system \((V, \mathcal{F})\) is called a **laminar** family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

Family is laminar if it has no two “properly intersecting” members: i.e., intersecting \(A \cap B \neq \emptyset\) and not comparable (one is not contained in the other).
Laminar Matroid

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A
B
A
B
A
B

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- Suppose we have a laminar family \(\mathcal{F}\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in \mathcal{F}\).
We can define a matroid with structures richer than just partitions.

A set system \((V,F)\) is called a **laminar** family if for any two sets \(A,B \in F\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

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Suppose we have a laminar family \(F\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in F\).

Then \((V,I)\) defines a matroid where

\[
I = \{ I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in F \} \tag{5.25}
\]
Laminar Matroid

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![Diagram of laminar family](image)

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\[
\mathcal{I} = \{ I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F} \}
\] (5.25)

- **Exercise:** what is the rank function here?