Logistics

Review

Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.
Let \((E, I)\) be an independence system, and we are given a non-negative modular weight function \(w : E \to \mathbb{R}_+\).

**Algorithm 1:** The Matroid Greedy Algorithm

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X\) s.t. \(X \cup \{v\} \in I\) do
3. \(v \in \arg \max \{w(v) : v \in E \setminus X, X \cup \{v\} \in I\}\);
4. \(X \leftarrow X \cup \{v\}\);

Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

**Theorem 9.2.2**

Let \((E, I)\) be an independence system. Then the pair \((E, I)\) is a matroid if and only if for each weight function \(w \in \mathbb{R}_+^E\), Algorithm 1 leads to a set \(I \in \mathcal{I}\) of maximum weight \(w(I)\).

Matroid Polyhedron in 2D

\[
P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \tag{9.10}
\]

Consider this in two dimensions. We have equations of the form:

\[
x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{9.11}
\]

\[
x_1 \leq r(\{v_1\}) \tag{9.12}
\]

\[
x_2 \leq r(\{v_2\}) \tag{9.13}
\]

\[
x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{9.14}
\]

Because \(r\) is submodular, we have

\[
r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{9.15}
\]

so since \(r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})\), the last inequality is either touching or active.
Matroid Polyhedron in 2D

And, if v2 is a loop ...

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I$.
- Taking the convex hull, we get the independent set polytope, that is
  \[ P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \]  
  (9.10)

- Since $\{1_I : I \in \mathcal{I}\} \subseteq P_{\text{ind. set}}$, we have
  $\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^T x : x \in P_{\text{ind. set}}\}$.
- Now take the rank function $r$ of $M$, and define the following polyhedron:
  \[ P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \]  
  (9.11)

- Now, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this next.
If \( x \in P_{\text{ind. set}} \), then
\[
x = \sum_{i} \lambda_i 1_{I_i}
\] (9.10)
for some appropriate vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \).

Clearly, for such \( x \), \( x \geq 0 \).

Now, for any \( A \subseteq E \),
\[
x(A) = x^T 1_A = \sum_{i} \lambda_i 1_{I_i}^T 1_A \leq \sum_{i} \lambda_i \max_{j: I_j \subseteq A} 1_{I_j}(E)
\] (9.12)
\[
= \max_{j: I_j \subseteq A} 1_{I_j}(E)
\] (9.13)
\[
= r(A)
\] (9.14)

Thus, \( x \in P^+_r \) and hence \( P_{\text{ind. set}} \subseteq P^+_r \).

So recall from a moment ago, that we have that
\[
P_{\text{ind. set}} = \text{conv} \{ \cup_{I \in \mathcal{I}} \{1_I\} \}
\subseteq P^+_r = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \tag{9.19}
\]

In fact, the two polyhedra are identical (and thus both are polytopes).
We’ll show this in the next few theorems.
Theorem 9.2.6

Let $M = (V, I)$ be a matroid, with rank function $r$, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \quad (9.19)$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i 1_{U_i} \quad (9.20)$$

Linear Program LP

Consider the linear programming primal problem

$$\text{maximize} \quad w^\top x$$
$$\text{subject to} \quad x_v \geq 0 \quad (v \in V) \quad (9.1)$$
$$x(U) \leq r(U) \quad (\forall U \subseteq V)$$

And its convex dual (note $y \in \mathbb{R}_+^{2^n}$, $y_U$ is a scalar element within this exponentially big vector):

$$\text{minimize} \quad \sum_{U \subseteq V} y_U r(U),$$
$$\text{subject to} \quad y_U \geq 0 \quad (\forall U \subseteq V)$$
$$\sum_{U \subseteq V} y_U 1_U \geq w \quad (9.2)$$

Thanks to strong duality, the solutions to these are equal to each other.
Consider the linear programming primal problem

\[
\begin{align*}
\text{maximize} & \quad w^\top x \\
\text{subject to} & \quad x_v \geq 0 \quad (v \in V) \\
& \quad x(U) \leq r(U) \quad (\forall U \subseteq V)
\end{align*}
\]  

\[ (9.3) \]

This is identical to the problem

\[
\text{max } w^\top x \text{ such that } x \in P_{\text{ind. set}}^+
\]

\[ (9.4) \]

where, again, \( P_{\text{ind. set}}^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \).

Therefore, since \( P_{\text{ind. set}} \subseteq P_{\text{ind. set}}^+ \), the above problem can only have a larger solution. I.e.,

\[
\text{max } w^\top x \text{ s.t. } x \in P_{\text{ind. set}} \leq \text{max } w^\top x \text{ s.t. } x \in P_{\text{ind. set}}^+.
\]

\[ (9.5) \]

Hence, we have the following relations:

\[
\max \{ w(I) : I \in \mathcal{I} \} \leq \max \{ w^\top x : x \in P_{\text{ind. set}} \} \leq \max \{ w^\top x : x \in P_{\text{ind. set}}^+ \}
\]

\[ (9.6) \]

\[ (9.7) \]

\[
\alpha_{\text{min}} \overset{\text{def}}{=} \min \left\{ \sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U 1_U \geq w \right\}
\]

\[ (9.8) \]

Theorem 8.6.1 states that

\[
\max \{ w(I) : I \in \mathcal{I} \} = \sum_{i=1}^{n} \lambda_i r(U_i)
\]

\[ (9.9) \]

for the chain of \( U_i \)'s and \( \lambda_i \geq 0 \) that satisfies \( w = \sum_{i=1}^{n} \lambda_i 1_{U_i} \) (i.e., the r.h.s. of Eq. 9.9 is feasible w.r.t. the dual LP).

Therefore, we also have

\[
\max \{ w(I) : I \in \mathcal{I} \} \geq \alpha_{\text{min}}
\]

\[ (9.10) \]
Polytope equivalence

- Hence, we have the following relations:
  \[
  \max \{ w(I) : I \in \mathcal{I} \} = \max \{ w^T x : x \in P_{\text{ind. set}} \} = \max \{ w^T x : x \in P_r^+ \}
  \]
  \[
  = \max \{ w^T x : x \in P_r^+ \}
  \]
  \[
  \def \alpha_{\text{min}} = \min \left\{ \sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w \right\}
  \]

- Therefore, all the inequalities above are equalities.
- And since \( w \in \mathbb{R}_E^+ \) is an arbitrary direction into the positive orthant, we see that \( P_r^+ = P_{\text{ind. set}} \)
- That is, we have just proven:

**Theorem 9.3.1**

\[
P_r^+ = P_{\text{ind. set}}
\] (9.11)

Polytope Equivalence (Summarizing the above)

- For each \( I \in \mathcal{I} \) of a matroid \( M = (E, \mathcal{I}) \), we can form the incidence vector \( \mathbf{1}_I \).
- Taking the convex hull, we get the independent set polytope, that is
  \[
P_{\text{ind. set}} = \text{conv} \{ \bigcup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \}
  \] (9.12)

- Now take the rank function \( r \) of \( M \), and define the following polyhedron:
  \[
P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \}
  \] (9.13)

**Theorem 9.3.2**

\[
P_r^+ = P_{\text{ind. set}}
\] (9.14)
Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 9.6, the LP problem with exponential number of constraints \( \max \{w^\top x : x \in P^+_r\} \) is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

**Theorem 9.3.3**

The LP problem \( \max \{w^\top x : x \in P^+_r\} \) can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since \( P^+_r \) is described as the intersection of an exponential number of half spaces).
- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

\[
\begin{align*}
  x &\geq 0 \\
  x(A) &\leq r(A) \quad \forall A \subseteq V \quad (9.15) \\
  x(V) & = r(V) \quad (9.17)
\end{align*}
\]

- Note the third requirement, \( x(V) = r(V) \).
- By essentially the same argument as above (Exercise:), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 9.15- 9.17 above.
- What does this look like?
Recall, a set $A$ is spanning in a matroid $M = (E, \mathcal{I})$ if $r(A) = r(E)$. Consider convex hull of incidence vectors of spanning sets of a matroid $M$, and call this $P_{\text{spanning}}(M)$.

**Theorem 9.3.4**

The spanning set polytope is determined by the following equations:

\begin{align*}
0 &\leq x_e \leq 1 \quad \text{for } e \in E \tag{9.18} \\
x(A) &\geq r(E) - r(E \setminus A) \quad \text{for } A \subseteq E \tag{9.19}
\end{align*}

Example of spanning set polytope in 2D.

Proof.

Recall that any $A$ is spanning in $M$ iff $E \setminus A$ is independent in $M^*$ (the dual matroid).

For any $x \in \mathbb{R}^E$, we have that

\[ x \in P_{\text{spanning}}(M) \iff 1 - x \in P_{\text{ind. set}}(M^*) \tag{9.20} \]

as we show next . . .
... proof continued.

- This follows since if \( x \in P_{\text{spanning}}(M) \), we can represent \( x \) as a convex combination:
  \[
  x = \sum_{i} \lambda_i 1_{A_i}
  \]  
  \[\text{(9.21)}\]

  where \( A_i \) is spanning in \( M \).

- Consider
  \[
  1 - x = 1_E - x = 1_E - \sum_i \lambda_i 1_{A_i} = \sum_i \lambda_i 1_{E \setminus A_i},
  \]  
  \[\text{(9.22)}\]

  which follows since \( \sum_i \lambda_i 1 = 1_E \), so \( 1 - x \) is a convex combination of independent sets in \( M^* \) and so \( 1 - x \in P_{\text{ind. set}}(M^*) \).

... which means, from the definition of \( P_{\text{ind. set}}(M^*) \), that

\[
1 - x \geq 0 \quad (9.23)
\]

\[
1_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \quad (9.24)
\]

And we know the dual rank function is

\[
r_{M^*}(A) = |A| + r_M(E \setminus A) - r_M(E) \quad (9.25)
\]

- giving

\[
x(A) \geq r_M(E) - r_M(E \setminus A) \text{ for all } A \subseteq E \quad (9.26)
\]
Matroids
where are we going with this?

- We've been discussing results about matroids (independence polytope, etc.).
- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it it worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...

Maximal points in a set

- Regarding sets, a subset $X$ of $S$ is a maximal subset of $S$ possessing a given property $\mathcal{P}$ if $X$ possesses property $\mathcal{P}$ and no set properly containing $X$ (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathcal{P}$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector $x$ is maximal within $P$ if it is the case that for any $\epsilon > 0$, and for all $e \in E$, we have that

$$x + \epsilon 1_e \notin P \quad (9.27)$$

- Examples of maximal regions (in red)
Maximal points in a set

- Regarding sets, a subset $X$ of $S$ is a maximal subset of $S$ possessing a given property $\mathcal{P}$ if $X$ possesses property $\mathcal{P}$ and no set properly containing $X$ (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathcal{P}$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector $x$ is maximal within $P$ if it is the case that for any $\epsilon > 0$, and for all $e \in E$, we have that
  \[ x + \epsilon 1_e \notin P \]  
  \[(9.27)\]

- Examples of non-maximal regions (in green)

![Examples of non-maximal regions](image)

Review

- The next slide comes from Lecture 5.
Matroids, independent sets, and bases

- **Independent sets:** Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called **independent** if $A \in \mathcal{I}$ and otherwise $A$ is called **dependent**.

- **A base of $U \subseteq E$:** For $U \subseteq E$, a subset $B \subseteq U$ is called a **base** of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

- **A base of a matroid:** If $U = E$, then a “base of $E$” is just called a **base** of the matroid $M$ (this corresponds to a **basis** in a linear space).

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**P-basis of $x$ given compact set $P \subseteq \mathbb{R}^E_+$**

**Definition 9.4.1 (subvector)**

$y$ is a **subvector** of $x$ if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

**Definition 9.4.2 (P-basis)**

Given a compact set $P \subseteq \mathbb{R}^E_+$, for any $x \in \mathbb{R}^E_+$, a subvector $y$ of $x$ is called a **$P$-basis** of $x$ if $y$ maximal in $P$.

In other words, $y$ is a $P$-basis of $x$ if $y$ is a maximal $P$-contained subvector of $x$.

Here, by $y$ being “maximal”, we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon 1_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of $y$ (the properties of $y$ being: in $P$, and a subvector of $x$).

In still other words: $y$ is a $P$-basis of $x$ if:

1. $y \leq x$ ($y$ is a subvector of $x$); and
2. $y \in P$ and $y + \epsilon 1_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ ($y$ is maximal $P$-contained).


A vector form of rank

- Recall the definition of rank from a matroid \( M = (E, I) \).

  \[
  \text{rank}(A) = \max \{ |I| : I \subseteq A, I \in \mathcal{I} \} \tag{9.28}
  \]

- **vector rank:** Given a compact set \( P \subseteq \mathbb{R}^E_{+} \), we can define a form of “vector rank” relative to this \( P \) in the following way: Given an \( x \in \mathbb{R}^E \), we define the vector rank, relative to \( P \), as:

  \[
  \text{rank}(x) = \max (y(E) : y \leq x, y \in P) \tag{9.29}
  \]

  where \( y \leq x \) is componentwise inequality (\( y_i \leq x_i, \forall i \)).

- If \( B_x \) is the set of \( P \)-bases of \( x \), then \( \text{rank}(x) = \max_{y \in B_x} y(E) \).

- If \( x \in P \), then \( \text{rank}(x) = x(E) \) (\( x \) is its own unique self \( P \)-basis).

- In general, this might be hard to compute and/or have ill-defined properties. We next look at an object that restrains and cultivates this form of rank.

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Polymatroidal polyhedron (or a “polymatroid”)

**Definition 9.4.3 (polymatroid)**

A *polymatroid* is a compact set \( P \subseteq \mathbb{R}^E_{+} \) satisfying

1. \( 0 \in P \)
2. If \( y \leq x \in P \) then \( y \in P \) (called **down monotone**).
3. For every \( x \in \mathbb{R}^E_{+} \), any maximal vector \( y \in P \) with \( y \leq x \) (i.e., any \( P \)-basis of \( x \)), has the same component sum \( y(E) \)

- **Condition 3 restated:** That is for any two distinct maximal vectors \( y^1, y^2 \in P \), with \( y^1 \leq x \) & \( y^2 \leq x \), with \( y^1 \neq y^2 \), we must have \( y^1(E) = y^2(E) \).

- **Condition 3 restated (again):** For every vector \( x \in \mathbb{R}^E_{+} \), every maximal independent subvector \( y \) of \( x \) has the same component sum \( y(E) = \text{rank}(x) \).

- **Condition 3 restated (yet again):** All \( P \)-bases of \( x \) have the same component sum.
Polymatroidal polyhedron (or a “polymatroid”)

**Definition 9.4.3 (polymatroid)**

A **polymatroid** is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$

- Vectors within $P$ (i.e., any $y \in P$) are called independent, and any vector outside of $P$ is called dependent.
- Since all $P$-bases of $x$ have the same component sum, if $B_x$ is the set of $P$-bases of $x$, than $\text{rank}(x) = y(E)$ for any $y \in B_x$.

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Matroid and Polymatroid: side-by-side

A **Matroid** is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set $I$ in $\mathcal{I}$, bounded by another set $A$, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A **Polymatroid** is:

1. a compact set $P \subseteq \mathbb{R}^E_+$
2. zero containing, $0 \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector $y$ in $P$, bounded by another vector $x$, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).
Polymatroidal polyhedron (or a “polymatroid”)

Left: $\exists$ multiple maximal $y \leq x$
Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 = \text{const}$.
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such $y$ must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E), \forall y$ is vacuous.

$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then $x$ is its own $P$-basis, i.e., it is a self $P$-basis.
- In a matroid, a base of $A$ is the maximally contained independent set. If $A$ is already independent, then $A$ is a self-base of $A$ (as we saw in Lecture 5).
Polymatroid as well? no

Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such $y$ must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2$, we see this is not a polymatroid.

- On the right, we have a similar situation, just the set of potential values that must have the $y(E)$ condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?
It appears that we have three possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

1. On the left: full dependence between \( v_1 \) and \( v_2 \)
2. In the middle: full independence between \( v_1 \) and \( v_2 \)
3. On the right: partial independence between \( v_1 \) and \( v_2 \)

- The \( P \)-bases (or single \( P \)-base in the middle case) are as indicated.
- Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
- The set of \( P \)-bases for a polytope is called the base polytope.

Polymatroidal polyhedron (or a “polymatroid”)

- Note that if \( x \) contains any zeros (i.e., suppose that \( x \in \mathbb{R}^E_+ \) has \( E \setminus S \) s.t. \( x(E \setminus S) = 0 \), so \( S \) indicates the non-zero elements, or \( S = \text{supp}(x) \)), then this also forces \( y(E \setminus S) = 0 \), so that \( y(E) = y(S) \). This is true either for \( x \in P \) or \( x \notin P \).
- Therefore, in this case, it is the non-zero elements of \( x \), corresponding to elements \( S \) (i.e., the support \( \text{supp}(x) \) of \( x \)), determine the common component sum.
- For the case of either \( x \notin P \) or right at the boundary of \( P \), we might give a “name” to this component sum, lets say \( f(S) \) for any given set \( S \) of non-zero elements of \( x \). We could name \( \epsilon \text{rank}(\frac{1}{\epsilon} 1_S) \triangleq f(S) \) for \( \epsilon \) very small. What kind of function might \( f \) be?
Polymatroid function and its polyhedron.

**Definition 9.4.4**

A **polymatroid function** is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P_f^+$ associated with a polymatroid function as follows

$$P_f^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \}$$  \hspace{1cm} (9.30)

$$= \{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \}$$  \hspace{1cm} (9.31)

Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0$$  \hspace{1cm} (9.33)

$$x_1 \leq f(\{v_1\})$$  \hspace{1cm} (9.34)

$$x_2 \leq f(\{v_2\})$$  \hspace{1cm} (9.35)

$$x_3 \leq f(\{v_3\})$$  \hspace{1cm} (9.36)

$$x_1 + x_2 \leq f(\{v_1, v_2\})$$  \hspace{1cm} (9.37)

$$x_2 + x_3 \leq f(\{v_2, v_3\})$$  \hspace{1cm} (9.38)

$$x_1 + x_3 \leq f(\{v_1, v_3\})$$  \hspace{1cm} (9.39)

$$x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\})$$  \hspace{1cm} (9.40)
Consider the asymmetric graph cut function on the simple chain graph \( v_1 - v_2 - v_3 \). That is, \( f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}| \) is count of any edges within \( S \) or between \( S \) and \( V \setminus S \), so that \( \delta(S) = f(S) + f(V \setminus S) - f(V) \) is the standard graph cut.

Observe: \( P_f^+ \) (at two views):

which axis is which?
**Matroid Polytopes**

**Polymatroid**

Associated polyhedron with a polymatroid function

- Consider modular function $w : V \rightarrow \mathbb{R}_+$ as $w = (1, 1.5, 2)^\top$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: $P^+_f$ (at two views):

![Polyhedron Diagram](image)

- which axis is which?

**Matroid Polytopes**

**Polymatroid**

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$. Alternatively, consider concavity violation, $1 = g(1 + 1) - g(1) < g(2 + 1) - g(2) = 1.5$.
- Observe: $P^+_f$ (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.
Summarizing the above, we have:

- Given a **polymatroid function** \( f \), its associated polytope is given as
  \[
  P_f^+ = \left\{ y \in \mathbb{R}_E^+ : y(A) \leq f(A) \text{ for all } A \subseteq E \right\}
  \] (9.41)

- We also have the definition of a **polymatroidal polytope** \( P \) (compact subset, zero containing, down-monotone, and \( \forall x \) any maximal independent subvector \( y \leq x \) has same component sum \( y(E) \)).

- Is there any relationship between these two polytopes?

- In the next theorem, we show that any \( P_f^+ \)-basis has the same component sum, when \( f \) is a polymatroid function, and \( P_f^+ \) satisfies the other properties so that \( P_f^+ \) is a polymatroid.

**Theorem 9.4.5**

*Let \( f \) be a polymatroid function defined on subsets of \( E \). For any \( x \in \mathbb{R}_E^+ \), and any \( P_f^+ \)-basis \( y^x \in \mathbb{R}_E^+ \) of \( x \), the component sum of \( y^x \) is

\[
y^x(E) = \text{rank}(x) = \max \left( y(E) : y \leq x, y \in P_f^+ \right)
= \min \left( x(A) + f(E \setminus A) : A \subseteq E \right)
\] (9.42)

As a consequence, \( P_f^+ \) is a polymatroid, since r.h.s. is constant w.r.t. \( y^x \).*

By taking \( B = \text{supp}(x) \) (so elements \( E \setminus B \) are zero in \( x \)), and for \( b \in B \), \( x(b) \) is big enough, the r.h.s. min has solution \( A^* = E \setminus B \). We recover submodular function from the polymatroid polyhedron via the following:

\[
f(B) = \max \left\{ y(B) : y \in P_f^+ \right\}
\] (9.43)

In fact, we will ultimately see a number of important consequences of this theorem (other than just that \( P_f^+ \) is a polymatroid).